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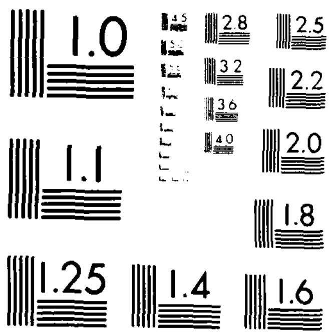

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On a Boundary Data Operator and Generalized Exterior Robin Problems for the Helmholtz Equation

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This report deals with boundary-value problems for the equation $\Delta u + \kappa^2 u = 0$ in an exterior domain Ω, in euclidean three-space, with a boundary condition of the form $\partial u/\partial \nu + B(u|\Gamma) = g$, $\Gamma = \partial\Omega$, is smooth, ν is the unit normal for Γ, $g \in L_2(\Gamma)$, and B is a bounded linear operator in $L_2(\Gamma)$ such that $i\zeta B$ is dissipative for some ζ lying in a certain set depending upon κ. It is required that the Neumann data $\partial u/\partial \nu$ and Dirichlet data $u|\Gamma$ in $L_2(\Gamma)$ be taken on in the normal-L_2 sense. The study is based upon the boundary-data operator A in $L_2(\Gamma)$, mapping $\partial u/\partial \nu$ to $u|\Gamma$ for appropriate outgoing solutions u in Ω. By studying the operator $I + BA$, it is proven that the problem is well-posed, and various construction techniques are established.

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ON A BOUNDARY-DATA OPERATOR AND GENERALIZED EXTERIOR ROBIN PROBLEMS FOR THE HELMHOLTZ EQUATION

1. INTRODUCTION

1.1 Orientation

We shall examine boundary-value problems for the Helmholtz, or reduced wave, equation $\Delta u + \kappa^2 u = 0$ in an exterior domain Ω_+ in $\mathbb{R}^3$, under the Sommerfeld radiation condition and a boundary condition of the form

$$\frac{\partial u}{\partial \nu} + B(u|\Gamma) = g . \tag{1.1}$$

Here, g is a specified element of the Hilbert space $H_0 := L_2(\Gamma)$, $\Gamma := \partial\Omega_+$, $B: H_0 \rightarrow H_0$ is a given linear operator, and $\partial u/\partial \nu$ and $u|\Gamma$ in H_0 denote, respectively, the normal derivative and trace of u on Γ in a "normal-L_2 sense" already employed by various investigators. Always supposing κ to be complex and nonzero with $Im\,\kappa \geqslant 0$, we shall require boundedness of B and dissipativeness of $i\,\overline{\zeta_\kappa^B}\,B$, *i.e.*,

$$Re \int_\Gamma i\,\overline{\zeta_\kappa^B}\,Bf \cdot \bar{f}\, d\Gamma \leqslant 0, \quad f \in H_0 , \tag{1.2}$$

for some ζ_κ^B that must lie in a certain complex set Z_κ constructed from κ. In particular, it will be seen that Z_κ always contains κ itself, so that the operator of multiplication by a function σ in $L_\infty(\Gamma)$ satisfying $Im(\bar{\kappa}\sigma) \geqslant 0$ is an example of a B that is acceptable within the present framework. The latter special case constitutes the more usual Robin, or impedance, or third, boundary condition that has already been studied in various settings. Thus, Leis [1] obtained uniqueness and existence results for continuous data g, with a real continuous σ taken to be nonpositive when $Im\,\kappa > 0$. Under the more general hypotheses that σ be complex and continuous, with $Im(\bar{\kappa}\sigma) \geqslant 0$, and still for continuous g, the corresponding assertions are developed in the book of Colton and Kress [2]. Angell and Kleinman [3,4], gave the first discussions of the Robin problem under weaker conditions on g and σ, requiring that the boundary condition be fulfilled in the normal-L_2 sense (our terminology) adopted here. Specifically, in Ref. 3 it was assumed that g and σ are in $L_\infty(\Gamma)$, while the inclusion $g \in H_0$ was allowed in Ref. 4, retaining the hypothesis $Im(\bar{\kappa}\sigma) \geqslant 0$ throughout. There appear to be some errors in the former work, most notably in the proof of Ref. 3, Theorem 3.6, while the results of Ref. 4 rest upon those in Ref. 3, this gap can be filled by using the results of Kersten [5] if the boundary manifold in Ref. 3 is supposed to be of class C^2. Finally, Angell and Kress study in Ref. 6 a boundary-integral reformulation of the exterior Robin problem that differs from that of Ref. 3, again for $g \in H_0$, $\sigma \in L_\infty(\Gamma)$, and $Im(\bar{\kappa}\sigma) \geqslant 0$. At any rate, the present study of exterior problems with boundary condition described by (1.1) and (1.2) appears to be a considerable generalization of those undertaken previously. Indeed, the condition here can be of a global type when B is, for example, an integral operator.

Our strategy consists of replacing the original problem with one involving an operator in H_0. In this respect, the present approach does not differ from those cited, but the method employed here in

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the development of the reformulation is new. All of our reasoning is based upon the "boundary-data" operator $A : H_0 \to H_0$ that maps Neumann data to the corresponding Dirichlet data for appropriate outgoing solutions of the Helmholtz equation in Ω_+. Using the well-known existence result for the classical exterior Neumann problem, we prove in §6 the existence of the compact operator A in H_0 which performs just this mapping, and we identify various of its properties. In particular, it turns out that $-i\zeta A$ is "strictly dissipative" whenever ζ lies in the set Z_κ, *i.e.*,

$$Re\int_\Gamma (-i\zeta Af)\cdot \bar{f}\, d\Gamma < 0\ ,\ f\in H_0\ ,\ f\neq 0\ ,\ \zeta\in Z_\kappa\ . \tag{1.3}$$

The combination of (1.3) and the postulated property (1.2) then leads to the assertion that $I+BA : H_0 \to H_0$ is injective, and so possesses a bounded inverse defined on H_0, whence the existence and continuous-dependence results for the original problem are shown to ensue. Uniqueness also follows from this circumstance but is proven directly in §3 (using (1.2)), where it is also shown that any solution must possess the "Green's-Theorem-type" representation in terms of its Dirichlet and Neumann data. In fact, the statements of §3, involving a generalization of the classical divergence theorem quite similar to that given in Ref. 3, are fundamental for the entire construction. Especially important is Corollary 3.5, extending results in Refs. 1 and 2, since it enables us to secure (1.3) and provides the basis for the definition of the set Z_κ given in §2, cf., also, the remarks preceding Corollary 3.5.

We must note that we require Γ to be of class C^2, *i.e.*, we consider only smooth boundaries; the compactness of A arises from this hypothesis. It is interesting to consider how the reasoning would change if Γ were supposed to be, say, only piecewise smooth. There, one would expect to be able to show that $I + BA$ is injective and Fredholm of index zero, assuming that the basic reasoning could be carried over in some appropriate manner.

Following the requisite comments on notation, in the remainder of this section, we proceed in §2 to the formulation of the generalized exterior Robin problem, including the description of the normal-L_2 manner in which boundary data are to be taken on. As noted, §3 is devoted to securing an extension of the classical divergence theorem and its consequences. A short catalogue of the properties of single- and double-layer potentials and associated integral operators appears in §4. In §5, we establish notation for a certain distinguished family of radiating-wave functions in Ω_+, and point out the existence of such families. The boundary-data operator A is studied in §6. The existence and continuous-dependence results are stated and proven in §7, following which, in §8, we take up the question of generating further results upon which one can base numerical algorithms for the computation of the solution. It is to be observed that each of the schemes presented is viable without regard for the particular value of κ. In §9 we remark on the implications for the exterior Dirichlet problem, and in §10 we conclude by indicating various directions in which one might attempt to extend the reasoning to other problems. In particular, the final section contains an existence, uniqueness, and continuous-dependence result that is superior to that of §7, and it is indicated that the arguments of §8 can be carried over.

1.2 Notation

The standard set-theoretic notations are employed. For example, if U and V are sets with $U \subset V$, then $V \sim U$ denotes the complement of U taken with respect to V.

When the range of function g lies in the domain of function f, we denote the resulting composite map by $f \circ g$. The function which is the restriction of a function f to a subset U of its domain is indicated by either $f \mid U$ or $f\mid_U$.

For a subset U of a topological space, U^0, U^-, and ∂U denote, respectively, the interior, closure, and boundary of U.

The standard inner product of the elements x and y in $\mathbb{C}^3$ shall be denoted by $x \bullet y$, the corresponding norm of x by $|x|$. It should cause no confusion to let $|\zeta|$ also stand for the modulus of $\zeta \in \mathbb{C}$; $\bar{\zeta}$ denotes the conjugate of ζ. For the open ball $\{y \in \mathbb{R}^3 : |y-x| < a\}$ in $\mathbb{R}^3$, of radius $a > 0$ and centered at $x \in \mathbb{R}^3$, we write $B_a(x)$.

Let U be a subset of $\mathbb{R}^3$. $C(U)$ denotes the complex linear space of all complex-valued continuous functions on U. When U is compact, let it be understood that $C(U)$ is equipped with its *supremum*-norm topology, in the absence of a stipulation to the contrary. By $C_H(U)$, we shall mean the collection of bounded Holder-continuous elements of $C(U)$, *i.e.*, those bounded $f \in C(U)$ for which there exist $C_f > 0$ and $\beta_f > 0$ such that $|f(y) - f(x)| \leqslant C_f |y - x|^{\beta_f}$ for all $x, y \in U$. If $U \subset U_0 \subset \mathbb{R}^3$, f is defined only on U, and we write $f \in C(U_0)$, then we imply that $f = f_0|U$ for some $f_0 \in C(U_0)$, *i.e.*, that $f \in C(U)$ and possesses a (unique) continuous extension to U_0.

Let Ω be an open subset of $\mathbb{R}^3$, and let k be a positive integer. As usual, $C^k(\Omega)$ $[C^k_H(\Omega)]$ is the set of all elements of $C(\Omega)$ that possess in Ω all partial derivatives of orders not exceeding k and each of which lies in $C(\Omega)$ [resp., in $C_H(\Omega)$]. The first partial derivative of $f \in C^k(\Omega)$ with respect to the ith cartesian coordinate is denoted by $f_{,i}$; more generally, if $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a 3-index of order $\alpha_1 + \alpha_2 + \alpha_3 \leqslant k$, then the αth partial derivative of f is written $f_{,\alpha}$. Whenever $f \in C^k(\Omega)$ and Ω_0 satisfies $\Omega \subset \Omega_0 \subset \Omega$, by the inclusion $f \in C^k(\Omega_0)$ we shall mean that each partial derivative of f can be extended continuously to Ω_0.

Now, suppose that k is a positive integer and M is a compact two-dimensional manifold of class C^k in $\mathbb{R}^3$, cf., e.g., Ref. 7. For $1 \leqslant l \leqslant k$, the space $C^l(M)$ is defined in the standard manner as the collection of all $f: M \to \mathbb{C}$ such that $f \circ h^{-1} \in C^l(h(U))$ whenever U is a coordinate patch on M and $h: U \to \mathbb{R}^2$ is a coordinate function for U.

The preceding definitions have obvious generalizations for $k = \infty$ and also for $\mathbb{C}^3$-valued functions. For the latter, we use the notations $C(\Omega, \mathbb{C}^3), C^k(\Omega; \mathbb{C}^3)$, *etc.*, to indicate the corresponding space of functions. Thus, if $f \in C^1(\Omega)$, grad $f \in C(\Omega, \mathbb{C}^3)$ denotes the gradient of f. If $f \in C^1(M)$, then one can define a surface gradient Grad $f \in C(M, \mathbb{C}^3)$ as in Colton and Kress [2]. With these authors, we define $C^1_H(M)$ to be the set of all those $f \in C^1(M)$ for which Grad $f \in C_H(M; \mathbb{C}^3)$.

We should point out that, because of the compactness of M, one can show that $C^1(M) \subset C_H(M)$, while if M is of class C^2, then $C^2(M) \subset C^1_H(M)$.

Let $\mathbf{M}$ be a measure space, with positive measure μ. For $1 \leqslant p < \infty$, $L_p(\mathbf{M})$ denotes the collection of (equivalence classes of) complex μ-measurable functions f defined μ-a.e. on $\mathbf{M}$ and such that $\int_{\mathbf{M}} |f|^p \, d\mu < \infty$, while $L_\infty(\mathbf{M})$ denotes the collection of essentially bounded μ-measurable functions defined μ-a.e. on $\mathbf{M}$. Each of these spaces is to be provided with its usual norm, under which it is a Banach space. In particular, the Lebesgue measure on $\mathbb{R}^3$ shall be denoted by λ, and the Lebesgue measure (induced by λ) on the manifold M of class C^1 in $\mathbb{R}^3$ is written λ_M; for a development of the latter, one can consult Ref. 8.

Let H be a Hilbert space. The orthogonal complement of a set $U \subset H$ shall be written $U^\perp$. If L is a linear operator densely defined in H with range in H, its (Hilbert-space) adjoint is denoted by L^*; and its null space and range are denoted by $\mathcal{N}(L)$ and $\mathcal{R}(L)$, respectively. If L is bounded, its norm shall be denoted by $\|L\|$, $\|\cdot\|$ being the norm on H. The symbol l_2 indicates the usual Hilbert space of all complex sequences $(c_n)_{n=1}^{\infty}$ with $\sum_{n=1}^{\infty} |c_n|^2 < \infty$.

Finally, $\square$ marks the completion of a proof.

2. FORMULATION OF THE GENERALIZED EXTERIOR ROBIN PROBLEM

We begin by imposing conditions that are to remain in force throughout for the underlying data. Let Ω_- be a bounded subset of $\mathbf{R}^3$ that is regularly open (*i.e.*, Ω_- is the interior of its closure), and the boundary $\Gamma := \partial\Omega_-$ of which is a two-dimensional manifold of class C^2. Then Ω_- lies locally "on one side" of Γ, while the latter is oriented by the unit-normal field ν on Γ which is "exterior" relative to Ω_-, so that, for each $x \in \Gamma$, $x + s\nu(x)$ lies in the exterior region $\Omega_+ := \mathbf{R}^3 \backslash \Omega_-^-$ for all sufficiently small positive s. In fact, because of the regularity of Ω_- and Γ, one can show that there exists $s_o > 0$ such that

$$\left.\begin{array}{l} x+s\nu(x)\in\Omega_+, \\ x-s\nu(x)\in\Omega_-, \end{array}\right\} \text{ whenever } x\in\Gamma \text{ and } 0 < s < s_o; \tag{2.1}$$

cf., *e.g.*, Ref. 8. It is also required that Ω_+ be connected. Further consequences of the geometric assumptions shall be pointed out as they are needed.

By H_o we denote the Hilbert space $L_2(\Gamma)$ with the inner product $<\cdot,\cdot>$, and associated norm $||\cdot||$, given by

$$<f,g> := \int_\Gamma f\cdot\bar{g}\,d\lambda_\Gamma, \quad f,g\in L_2(\Gamma).$$

κ shall be a fixed nonzero complex number with $Im\,\kappa \geqslant 0$; if $Im\,\kappa = 0$, then we shall suppose that $\kappa > 0$. With κ we associate the subset $Z_\kappa \subset \mathbb{C}$ defined by

$$Z_\kappa := \begin{cases} \{\zeta\in\mathbb{C} \mid \zeta\in\mathbf{R},\, \zeta>0\} = (0,\infty), & \text{if } Im\,\kappa = 0, \\ \{\zeta\in\mathbb{C} \mid Im\,\zeta \geqslant 0,\, Im\,(\zeta\bar{\kappa}^2) \leqslant 0, & \\ \quad \text{and } \{Im\,\zeta\}^2 + \{Im\,(\zeta\bar{\kappa}^2)\}^2 > 0\}, & \text{if } Im\,\kappa > 0. \end{cases} \tag{2.2}$$

Clarifying remarks are in order here. The motivation for the definition of Z_κ will become clear as we proceed, although its ultimate origin is to be found in Corollary 3.5, *infra*. As examples, we point out that κ itself clearly always lies in Z_κ, while $\kappa^2 \in Z_\kappa$ when $Im\,\kappa = 0$, but if $Im\,\kappa > 0$, then $\kappa^2 \in Z_\kappa$ *iff* $Re\,\kappa > 0$ (if $Im\,\kappa > 0$, then $\kappa^2 \in Z_\kappa$ *iff* $Im\,(\kappa^2) > 0$, which happens in this case *iff* $Re\,\kappa > 0$). It will be important later to know that $Re\,(\zeta\bar{\kappa}) \geqslant 0$ whenever $\zeta \in Z_\kappa$, with strict inequality obtaining if $Im\,\kappa = 0$. Indeed, if the latter equality should hold, obviously $Re(\zeta\bar{\kappa}) = \zeta\kappa > 0$ for $\zeta \in Z_\kappa$. Now suppose that $Im\,\kappa > 0$ and $\zeta \in Z_\kappa$; then $Im\,\zeta \geqslant 0$ and

$$Im\,(\zeta\bar{\kappa}^2) = \{(Re\,\kappa)^2 - (Im\,\kappa)^2\}\cdot Im\,\zeta - 2\cdot Im\,\kappa\cdot Re\,\kappa\cdot Re\,\zeta \leqslant 0,$$

so

$$Im\,\kappa\cdot Im\,\zeta + 2\cdot Re\,\kappa\cdot Re\,\zeta \geqslant (Re\,\kappa)^2\cdot Im\,\zeta/Im\,\kappa \geqslant 0.$$

Thus, if $Re\,\kappa\cdot Re\,\zeta < 0$, we get

$$\begin{aligned} Re(\zeta\bar{\kappa}) &= Im\,\kappa\cdot Im\,\zeta + Re\,\kappa\cdot Re\,\zeta \\ &> Im\,\kappa\cdot Im\,\zeta + 2\cdot Re\,\kappa\cdot Re\,\zeta \geqslant 0, \end{aligned}$$

while if $Re\,\kappa\cdot Re\,\zeta \geqslant 0$, then certainly $Re(\zeta\bar{\kappa}) \geqslant 0$. This proves the assertion. It may be helpful to observe that the inclusion $\zeta \in Z_\kappa$ implies that $\zeta \neq 0$ and constitutes a condition on the argument of ζ relative to that of κ. In fact, if we write $\kappa = \kappa_o e^{i\theta_\kappa}$ ($\kappa_o > 0$, $0 \leqslant \theta_\kappa < \pi$) and $\zeta = \zeta_o e^{i\theta_\zeta}$ ($\zeta_o > 0$, $0 \leqslant \theta_\zeta \leqslant \pi$), then a bit of computation reveals that

$$\zeta \in Z_\kappa \textit{ iff } \max\{0, 2\theta_\kappa - \pi\} \leqslant \theta_\zeta \leqslant \min\{\pi, 2\theta_\kappa\} \text{ and}$$

$$\text{both inequalities are strict if } \theta_\kappa = \frac{\pi}{2}.$$

Returning to our listing of hypotheses, we finally suppose that there has been given a bounded linear operator $B: H_0 \to H_0$ (which may depend upon κ) with the property that

$$\text{there exists } \zeta_\kappa^B \in Z_\kappa \text{ with } Im < \overline{\zeta_\kappa^B} Bf, f > \geqslant 0 \text{ for all } f \in H_0. \tag{2.2$_2$}$$

We wish to examine problems generated by the Helmholtz equation

$$\Delta u + \kappa^2 u = 0 \tag{2.3}$$

in Ω_+, the Sommerfeld radiation condition

$$\lim_{\rho \to \infty} \rho \cdot \{\tau \bullet \text{grad } u(\rho\tau) - i\kappa u(\rho\tau)\} = 0 \text{ uniformly in } \tau \text{ for } |\tau| = 1, \tag{2.4}$$

and the boundary condition

$$\frac{\partial u}{\partial \nu} + B(u|\Gamma) = g,$$

in which g is a given element of H_0 and $\partial u / \partial \nu$, $u|\Gamma \in H_0$ are to be interpreted in some reasonable manner as, respectively, a normal derivative and trace of u on Γ. Before being more precise in the formulation of these problems, we must specify the sense in which these boundary-data functions are to be associated with an appropriate function u.

For this purpose, we define $N_s: \Gamma \to \mathbf{R}^3$ for any $s \in \mathbf{R}$ by setting

$$N_s(x) := x + s\nu(x), \quad x \in \Gamma . \tag{2.5}$$

Now, if Ω_1 is an open set containing Γ and $f \in C^1(\Omega_+ \cap \Omega_1)$, then $f \circ N_s$ and $\nu \bullet \{(\text{grad } f) \circ N_s\}$ are defined and continuous on Γ for all sufficiently small positive s, because of (2.1), so it is sensible to ask for the convergence of the associated function-valued maps, as $s \to 0^+$, in, say, either of the Banach spaces $C(\Gamma)$ or $L_p(\Gamma)$ $(p \geqslant 1)$. With this motivation, we introduce the following terminology.

2.1 Definitions

Let $\Omega_1 \subset \mathbf{R}^3$ be an open set containing Γ, and $p \in [1, \infty)$.

(i) Let $f \in C(\Omega_+ \cap \Omega_1)$ [resp., $f \in C(\Omega_+ \cap \Omega_1; \mathbb{C}^3)$]. If $\lim_{\epsilon \to 0^+} f \circ N_\epsilon$ [resp., $\lim_{\epsilon \to 0^+} \nu \bullet (f \circ N_\epsilon)$] exists in the $L_p(\Gamma)$-sense, then we shall say that f *has a normal trace on* Γ *in the* L_p*-sense*, and denote the limit by $f|_\Gamma^p$ [resp., $(\nu \bullet f)|_\Gamma^p$].

(ii) Let $f \in C^1(\Omega_+ \cap \Omega_1)$, so that grad $f \in C(\Omega_+ \cap \Omega_1; \mathbb{C}^3)$. If grad f has a normal trace on Γ in the L_p-sense, *i.e.*, if $\lim_{\epsilon \to 0^+} \nu \bullet \{(\text{grad } f) \circ N_\epsilon\}$ exists in the $L_p(\Gamma)$-sense, then we shall say that f *has a normal derivative on* Γ *in the* L_p*-sense*, and denote the limit by $f_{,\nu}^p$ $(= (\nu \bullet \text{grad } f)|_\Gamma^p)$. If this limit exists in the $L_\infty(\Gamma)$-sense, *i.e.*, in the $C(\Gamma)$-sense, then we shall say that f *has a normal derivative on* Γ *in the uniform sense*, and denote the limit by $f_{,\nu}^c$.

2.2 Remarks

Let Ω_Γ be as in §2.1 and $1 \leq p \leq q < \infty$.

(a) Let $f \in C(\Omega_+^- \cap \Omega_\Gamma)$: then it is easy to see that $\lim_{\epsilon \to 0^+} f \circ N_\epsilon$ exists in the uniform sense and equals $f|_\Gamma$, the restriction of f to Γ; by the compactness of Γ, then $f|_\Gamma^p$ exists and equals $f|_\Gamma$. The corresponding conclusions hold for vector-valued functions.

(b) Again using the compactness of Γ, we know that $C(\Gamma) \subset L_q(\Gamma) \subset L_p(\Gamma)$, with convergence in $C(\Gamma)$ implying convergence in $L_q(\Gamma)$ to the same limit, and convergence in $L_q(\Gamma)$ implying convergence in $L_p(\Gamma)$ to the same limit. Thus, for $f \in C(\Omega_+ \cap \Omega_\Gamma)$, the existence of $f|_\Gamma^q$ implies that of $f|_\Gamma^p$ and $f|_\Gamma^q = f|_\Gamma^p$; for $f \in C^1(\Omega_+ \cap \Omega_\Gamma)$, the existence of either $f,_\nu^C$ or $f,_\nu^q$ implies that of $f,_\nu^p$ and the equality of the limits. In the sequel, we shall generally use simple facts like these without comment.

(c) Let $f \in C^1(\Omega_+^- \cap \Omega_\Gamma)$, so that grad f in $\Omega_+ \cap \Omega_\Gamma$ possesses a continuous extension (grad f) to $\Omega_+^- \cap \Omega_\Gamma$. Just as in (a), we can conclude that the uniform limit $f,_\nu^C := \lim_{\epsilon \to 0^+} \nu \bullet \{(\mathrm{grad}\, f)^- \circ N_\epsilon\}$ exists and equals $\nu \bullet (\mathrm{grad}\, f)^-|_\Gamma$ (whence $f,_\nu^p$ also exists and equals $f,_\nu^C$). If we should have $f \in C^1(\Omega_\Gamma)$, then $f,_\nu^C = \nu \bullet (\mathrm{grad}\, f)|_\Gamma$, the "usual" expression; in this latter situation, we shall write $f,_\nu$ in place of $f,_\nu^C$.

(d) We could also consider functions defined in an Ω_--neighborhood of Γ, formulating definitions and making remarks analogous to those just set forth. We shall suppose that this has been done.

Now we introduce the sorts of functions in which we shall be most interested:

2.3 Definitions

(i) Let f be a complex function with domain in Ω_+ and containing $\Omega_+ \cap \Omega_f$ for some open Ω_f containing Γ. If the restriction of f to $\Omega_+ \cap \Omega_f$ is in $C^1(\Omega_+ \cap \Omega_f)$, then we shall say that f is L_2-*regular at* Γ *iff* both $f|_\Gamma^2$ and $f,_\nu^2$ exist; when f has this property, we shall refer to $f|_\Gamma^2$ and $f,_\nu^2$ as the *Dirichlet data* and the *Neumann data of* f, respectively. We make the same definitions when Ω_- replaces Ω_+ in the preceding statement.

(ii) The set $W(\Omega_+;\kappa)$ is the collection of all $u \in C^2(\Omega_+)$ that are L_2-regular at Γ, satisfy (2.3) in Ω_+, and for which (2.4) is true.

$W(\Omega_+;\kappa)$ is the family of outgoing waves in which we choose to search for solutions of the exterior boundary-value problem that we pose now.

2.4 The Generalized Exterior Robin Problem ER(g|B;κ):

Recall the hypotheses placed upon Ω_+, κ, and B. For $g \in H_0$, show that there exists precisely one corresponding $u_g \in W(\Omega_+;\kappa)$ such that

$$u_{g,\nu}^2 + Bu_g|_\Gamma^2 = g \,. \tag{2.6}$$

Certain terminology is convenient. Let $g \in C(\Gamma)$ and suppose that there exists the solution function u_g for $ER(g|B;\kappa)$. If $u_g \in C(\Omega_+^-)$ and $u_{g,\nu}^C$ exists, we shall refer to u_g as a *weakly classical solution function*, while if $u_g \in C^1(\Omega_+^-)$, we shall say that the solution function is *classical*. For any $g \in H_0$, $ER(g|0;\kappa)$ is termed the corresponding *exterior Neumann problem*. In this language, we can already assert that the exterior Neumann problem $ER(g|0;\kappa)$ is solvable and, moreover, the solution function is weakly classical, for any $g \in C(\Gamma)$, cf., *e.g.*, [2, Theorem 3.25, pursuant to the appropriate definitions on pp. 68 and 76].

In §3, we shall see that $ER(0 \mid B; \kappa)$ has only the trivial solution. In §7, it is shown that there exists a solution function for $ER(g \mid B; \kappa)$ whenever $g \in H_o$, and that the resultant solution mapping depends continuously upon g and B, in a sense to be made precise. The hypothesis (2.2) enables us to obtain the existence and uniqueness results in a very simple manner, once certain preliminaries have been established.

Careful distinction should be made between the normal trace on Γ in the L_2-sense, considered here, and the more familiar trace that is shown to exist for a function lying in an appropriate Sobolev space (cf., *e.g.*, [9]). In Refs. 10, 11, and 12, Mikhailov studies the Dirichlet problem for an elliptic operator with real coefficients in a domain such as Ω , satisfaction of the boundary condition being required in the L_2-normal-trace sense. He points out that such a formulation is more general than that in terms of the Sobolev space $H^1(\Omega\ )$, since the set of Sobolev traces on Γ of functions in the latter space does not even include $C(\Gamma)$, and so is a proper linear submanifold of $H_o := L_2(\Gamma)$. Further information concerning the history of the idea of the normal trace in the L_2-sense is to be found in Refs. 10, 11, and 13.

3. A GENERALIZATION OF THE DIVERGENCE THEOREM AND ITS COROLLARIES

Angell and Kleinman [3] give a generalization of the classical divergence theorem, for a function in $C^1(\Omega; \mathbb{C}^3)$, the components of which have normal traces on $\partial\Omega$ in the L_2-sense; there, $\Omega \subset \mathbb{R}^3$ is a bounded open set with $\partial\Omega$ of class $C^{1,1}$ and possessing "approximating parallel surfaces" in a certain sense (Theorem 3.5 of Ref. 3). However, this version of generalization does not suffice in the applications for which they intended it (*e.g.*, Corollaries 1 and 2 and Theorem 3.7 of Ref. 3). Required there is a statement in which the normal traces on $\partial\,\Omega$ exist in the L_1-sense, which can evidently be obtained under the hypotheses of Ref. 3.

Following the lead in Ref. 3, we shall give a slightly different generalization of the classical divergence theorem, based upon the idea of the normal trace for a vector-valued function. To facilitate the applications that we have in mind, we shall cast all statements in terms of Ω , and Ω , although it will be clear that we could have first derived results for any bounded open set possessing the same regularity as Ω .

Let us introduce some convenient notation: we set

$$\Omega_{\mp}^{\epsilon} := \{ x \in \Omega_{\mp} \mid \operatorname{dist}(x, \Gamma) := \inf_{y \in \Gamma} |y - x| > \epsilon \}, \quad \text{for } \epsilon > 0, \tag{3.1}$$

and

$$\Omega(R; x) := \Omega \cap B_R(x), \tag{3.2}$$

whenever $\Omega \subset \mathbb{R}^3$, $x \in \mathbb{R}^3$, and $R > 0$. Further, let

$$\rho_x(y) := \frac{y - x}{|y - x|}, \quad x, y \in \mathbb{R}^3,\ y \neq x. \tag{3.3}$$

Finally, we write

$$\iota_- = 1, \quad \iota_+ = 0. \tag{3.4}$$

3.1 Theorem

Recall the hypotheses placed upon Ω and Γ. Let $x_0 \in \mathbb{R}^3$ and $R > 0$ be chosen so that $\Omega \subset B_R(x_0)$

(i) Suppose that $f \in C^1(\Omega_\pm;\mathbb{C}^3)$ has a normal trace on Γ in the L_1-sense. Then

$$\lim_{\epsilon\to 0^+} \int_{\Omega_\pm^\epsilon(R;x_o)} \operatorname{div} f \, d\lambda = \mp\int_\Gamma (\nu\bullet f)|_\Gamma^\pm \, d\lambda_\Gamma + \iota_\pm \cdot \int_{\partial B_R(x_o)} \rho_{x_o}\bullet f \, d\lambda_{\partial B_R(x_o)} . \tag{3.5}$$

(ii) Let $x \in \Omega_\pm$, and suppose that $f \in C^1(\Omega_\pm\backslash\{x\};\mathbb{C}^3)$ has a normal trace on Γ in the L_1-sense. Then, if R is also such that $x \in B_R(x_o)$, and $\delta > 0$ is chosen so that $B_\delta(x)^- \subset \Omega_\pm(R;x_o)$,

$$\lim_{\epsilon\to 0^+} \int_{\Omega_\pm^\epsilon(R;x_o)\backslash B_\delta(x)^-} \operatorname{div} f \, d\lambda = \mp\int_\Gamma (\nu\bullet f)|_\Gamma^\pm \, d\lambda_\Gamma + \iota_\pm \int_{\partial B_R(x_o)} \rho_{x_o}\bullet f \, d\lambda_{\partial B_R(x_o)}$$

$$- \int_{\partial B_\delta(x)} \rho_x \bullet f \, d\lambda_{\partial B_\delta(x)} . \tag{3.6}$$

Proof: We shall prove only the statements corresponding to "+"; the reasoning for those corresponding to "−" does not essentially differ.

We begin by citing a number of facts that can be proven on the basis of the regularity assumptions concerning Ω_- and Γ; cf. [8]: there exists a positive ϵ_0 such that for $0 < \epsilon < \epsilon_0$,

(a) Ω_+^ϵ is a regularly open set, $\partial\Omega_+^\epsilon$ is a manifold of class C^1, $N_\epsilon(\Gamma) = \partial\Omega_+^\epsilon$, and $N_\epsilon\colon \Gamma \to \partial\Omega_+^\epsilon$ is a diffeomorphism;

(b) a unit normal to $\partial\Omega_+^\epsilon$ at $y \in \partial\Omega_+^\epsilon$, directed "inward" relative to Ω_+^ϵ, is given by $\nu(N_\epsilon^{-1}(y))$, and $\nu\circ N_\epsilon^{-1} \in C^1(\partial\Omega_+^\epsilon;\mathbb{C}^3)$; here, N_ϵ^{-1} denotes the inverse of $N_\epsilon\colon \Gamma \to \partial\Omega_+^\epsilon$;

(c) there exists a positive continuous function JN_ϵ on Γ (the "generalized Jacobian") such that if $h \in L_1(\partial\Omega_+^\epsilon)$, then $h\circ N_\epsilon \in L_1(\Gamma)$ and

$$\int_{\partial\Omega_+^\epsilon} h \, d\lambda_{\partial\Omega_+^\epsilon} = \int_\Gamma (h\circ N_\epsilon)\cdot JN_\epsilon \, d\lambda_\Gamma .$$

Moreover,

(d) $\lim_{\epsilon\to 0^+} JN_\epsilon = 1$ in $L_\infty(\Gamma)$, *i.e.*, uniformly on Γ.

There are corresponding statements for $N_{-\epsilon}$ replacing N_ϵ and Ω_-^ϵ replacing Ω_+^ϵ.

We proceed to the proofs.

(i) Clearly, we can suppose that $\epsilon_o > 0$ is such that (a), (b), and (c) are true and $\partial(\Omega_+^\epsilon(R;x_o)) = \partial\Omega_+^\epsilon \cup \partial B_R(x_o)$ for $0 < \epsilon < \epsilon_o$, since $\Omega_- \subset B_R(x_o)$. For any such ϵ, the classical statement of the divergence theorem can be applied to produce

$$\int_{\Omega_+^\epsilon(R;x_o)} \operatorname{div} f \, d\lambda = -\int_{\partial\Omega_+^\epsilon} (\nu\circ N_\epsilon^{-1})\bullet f|_{\partial\Omega_+^\epsilon} \, d\lambda_{\partial\Omega_+^\epsilon} + \int_{\partial B_R(x_o)} \rho_{x_o}\bullet f|_{\partial B_R(x_o)} \, d\lambda_{\partial B_R(x_o)} , \tag{3.7}$$

while (c) allows the transformation

$$\int_{\partial\Omega_+^\epsilon} (\nu\circ N_\epsilon^{-1})\bullet f|_{\partial\Omega_+^\epsilon} \, d\lambda_{\partial\Omega_+^\epsilon} = \int_\Gamma \nu\bullet(f\circ N_\epsilon)\cdot JN_\epsilon \, d\lambda_\Gamma .$$

Now, by hypothesis, $(\nu\bullet f)|_\Gamma^+ := \lim_{\epsilon\to 0^+} \nu\bullet(f\circ N_\epsilon)$ exists in the $L_1(\Gamma)$-sense, whence, in view of (d), we can conclude that

$$\lim_{\epsilon \to 0^+} \int_\Gamma \nu \bullet (f \circ N_\epsilon)\, JN_\epsilon \, d\lambda_\Gamma = \int_\Gamma (\nu \bullet f)|_\Gamma^1 \, d\lambda_\Gamma \,.$$

From the latter equality and (3.7), (3.5) follows (for the "+" case).

(ii) Assuming that R is such that $x \in B_R(x_o)$ (as well as $\Omega_-^- \subset B_R(x_o)$), then $B_\delta(x)^- \subset \Omega_+(R;x_o)$ for all sufficiently small positive δ. Fix such a δ. Then we have both $B_\delta(x)^- \subset \Omega_+^\epsilon(R;x_o)$ and $\partial(\Omega_+^\epsilon(R;x_o)\backslash B_\delta(x_o)^-) = \partial\Omega_+^\epsilon \cup \partial B_R(x_o) \cup \partial B_\delta(x_o)$ for all sufficiently small positive ϵ. When the classical divergence theorem is applied to f in $\Omega_+^\epsilon(R;x_o)\backslash B_\delta(x_o)^-$ for such ϵ and the reasoning in the proof of (i) is retraced, the result is (3.6) (for the "+" case). □.

From Theorem 3.1, we can proceed to derive extensions of the well-known consequences of the classical divergence theorem. Several of these will be given; their proofs involve such familiar arguments that we shall be as brief as possible in outlining the reasoning.

3.2 Corollary

Let $u \in C^2(\Omega_\pm)$ have a normal derivative on Γ in the L_2-sense, and $v \in C^1(\Omega_\pm)$ have a normal trace on Γ in the L_2-sense. Let $x_o \in \mathbb{R}^3$ and $R > 0$ be chosen so that $\Omega_-^- \subset B_R(x_o)$. Then

$$\lim_{\epsilon \to 0^+} \int_{\Omega_\pm^\epsilon(R;x_o)} \{v \cdot \Delta u + \text{grad}\, v \bullet \text{grad} u\}\, d\lambda$$

$$= \mp \int_\Gamma v|_\Gamma^2 \cdot u,_\nu^2 \, d\lambda_\Gamma + \iota_\pm \cdot \int_{\partial B_R(x_o)} v \cdot (\rho_{x_o} \bullet \text{grad} u)\, d\lambda_{\partial B_R(x_o)} \,. \tag{3.8}$$

Proof: Again, we consider only the "+" case. Let $f := v \,\text{grad}\, u$, so $f \in C^1(\Omega_+;\mathbb{C}^3)$. For all sufficiently small $\epsilon > 0$, we have $\nu \bullet (f \circ N_\epsilon) = (v \circ N_\epsilon)\, \nu \bullet \{(\text{grad}\, u) \circ N_\epsilon\}$, while $v|_\Gamma^2 := \lim_{\epsilon \to 0^+} v \circ N_\epsilon$ and $u,_\nu^2 := \lim_{\epsilon \to 0^+} \nu \bullet \{(\text{grad}\, u) \circ N_\epsilon\}$ exist in H_o. Thus, use of Hölder's inequality shows that

$$(\nu \bullet f)|_\Gamma^1 := \lim_{\epsilon \to 0^+} \nu \bullet (f \circ N_\epsilon) = v|_\Gamma^2 \cdot u,_\nu^2 \text{ in } L_1(\Gamma),$$

i.e., f has the normal trace $v|_\Gamma^2 \cdot u,_\nu^2$ on Γ in the L_1-sense. Now apply (3.5) to arrive at (3.8). □.

The next two results are stated for functions defined in Ω_+ but clearly have counterparts for functions defined in Ω_- (involving no radiation condition, of course).

3.3 Corollary

Let $u,v \in W(\Omega_+;\kappa)$. Then

$$\int_\Gamma \{ u|_\Gamma^2 \cdot v,_\nu^2 - u,_\nu^2 \cdot v|_\Gamma^2 \}\, d\lambda_\Gamma = 0 \,. \tag{3.9}$$

Proof. Since $Im\kappa \geq 0$, we know that $u(\rho\tau)$ and $v(\rho\tau)$ are $O(1/\rho)$ as $\rho \to \infty$, uniformly in τ for $|\tau| = 1$, following from the classical Green's-theorem representation for u and v in the exterior of a sufficiently large ball as an integral over the surface of the ball. Write (3.8), reverse the roles of u and v, and subtract the two equalities; with the fact just cited and the radiation condition fulfilled by u and v, the integral over $\partial B_R(x_o)$ must tend to zero as $R \to \infty$. This implies (3.9). □.

Now, we wish to show that each element of $W(\Omega_+;\kappa)$ can be represented, as in the classical case, by an integral taken over Γ, involving its Dirichlet and Neumann data and a fundamental solution for the operator $\Delta + \kappa^2$. For the latter, we shall take $E(\cdot;\cdot)$, given by

$$E(x;y) := -\frac{e^{i\kappa|y-x|}}{2\pi \cdot |y-x|}, \qquad x,y \in \mathbb{R}^3, x \neq y. \tag{3.10}$$

Let $E_{,j}$ and $E_{,\bar{j}}$ denote, respectively, the partial derivatives of E with respect to the jth Cartesian coordinate of its primary and secondary arguments. When $x \in \mathbb{R}^3$, we shall write E_x for the function $y \mapsto E(x;y)$ in $\mathbb{R}^3 \setminus \{x\}$. In keeping with these notations and those introduced already, we set

$$E_{x,\nu}(y) := \nu(y) \bullet \operatorname{grad} E_x(y) = \sum_{j=1}^{3} \nu_j(y) \cdot E_{,\bar{j}}(x;y), \quad x \in \mathbb{R}^3, y \in \Gamma \setminus \{x\},$$

and

$$E_{,\nu(x)}(x;y) := \sum_{j=1}^{3} \nu_j(x) \cdot E_{,j}(x;y), \qquad x \in \Gamma, y \in \mathbb{R}^3 \setminus \{x\}.$$

Of course, for each $x \in \mathbb{R}^3, E_x$ is a solution of (2.3) in $\mathbb{R}^3 \setminus \{x\}$ and satisfies (2.4); in particular, whenever $x \in \Omega$ it is clear that E_x (more precisely, $E_x | \Omega_+$) is an element of $W(\Omega_+;\kappa)$.

3.4 Corollary

Let $u \in W(\Omega_+;\kappa)$. Then

$$u(x) = \frac{1}{2} \int_\Gamma \{E_x \cdot u_{,\nu}^{\Sigma} - E_{x,\nu} \cdot u|_\Gamma^{\Sigma}\} \, d\lambda_\Gamma, \quad x \in \Omega_+. \tag{3.11}$$

Proof: Fix $x \in \Omega_+$. Starting from statement (ii) of Theorem 3.1, we reason as in the proofs of Corollaries 3.2 and 3.3 to arrive at the equality

$$\int_\Gamma \{E_x \, u_{,\nu}^{\Sigma} - E_{x,\nu} \cdot u|_\Gamma^{\Sigma}\} \, d\lambda_\Gamma = -\int_{\partial B_\delta(x)} \rho_x \bullet \{E_x \cdot \operatorname{grad} u - u \operatorname{grad} E_x\} \, d\lambda_{\partial B_\delta(x)}, \tag{3.12}$$

holding whenever $B_\delta(x)^- \subset \Omega_+$, *i.e.*, for all sufficiently small $\delta > 0$. In the usual manner, the limit, as $\delta \to 0^+$, of the expression on the right in (3.12) is found to be $2u(x)$, whence (3.11) clearly follows. $\Box$.

In Ref. 2, Theorem 3.12, it is proven that when $u \in C^2(\Omega_+) \cap C(\Omega_+^-)$ satisfies (2.3) in Ω_+ and (2.4), and possesses the uniform normal derivative $u_{,\nu}^{\zeta}$, then from the inequality $Im < \kappa u|_\Gamma, u_{,\nu}^{\zeta} > \geqslant 0$ must follow the vanishing of u. The uniqueness statements for the various exterior Dirichlet, Neumann, and Robin problems, as well as the transmission problem, considered in Ref. 2 are based upon this fact. Our next corollary subsumes this result, extending it in two directions. The extension is of particular importance for present purposes, since it will give the desired uniqueness result directly here also, while providing the key to the existence proof when $B \neq 0$. This corollary also extends an implication of a result of Leis [1, Lemma 6], the proof of which has suggested the reasoning that we are about to employ.

3.5 Corollary

Let $u \in W(\Omega_+;\kappa)$. If $Im < \zeta u|_\Gamma^{\Sigma}, u_{,\nu}^{\Sigma} > \geqslant 0$ for some $\zeta \in Z_\kappa$, then $u = 0$.

Proof: For brevity, we shall write $u_{,\rho} := \rho_0 \bullet \operatorname{grad} u$, $B_R := B_R(0)$, and $\Omega(R) := \Omega(R;0)$ $(\Omega \subset \mathbb{R}^3)$. Using Corollary 3.2, for all sufficiently large positive R, we have

$$\int_\Gamma u|_\Gamma^{\Sigma} \, \bar{u}_{,\nu}^{\Sigma} \, d\lambda_\Gamma = \lim_{\epsilon \to 0^+} \int_{\Omega_\epsilon^+(R)} \{\kappa^2 |u|^2 - |\operatorname{grad} u|^2\} \, d\lambda + \int_{\partial B_R} u \cdot \bar{u}_{,\rho} \, d\lambda_{\partial B_R}.$$

so, for $\zeta \in \mathbb{C}$,

$$Im < \zeta u|_\Gamma^2 , u_{,\nu}^2 > = - \lim_{\epsilon \to 0^+} \int_{\Omega_+^\epsilon(R)} \{ - Im(\zeta \bar{\kappa}^2)|u|^2 + Im\zeta \cdot |\text{grad } u|^2 \} \, d\lambda$$

$$+ Im \int_{\partial B_R} \zeta u \, \overline{u_{,\rho}} \, d\lambda_{\partial B_R} .$$

Now suppose that $\zeta \in Z_\kappa$; then the integrand figuring in the limit on the right is nonnegative, whence an application of B. Levi's Theorem [14, Theorem 12.22] allows the conclusion that the (finite) limit is the integral over $\Omega_+(R)$. Moreover, using the radiation condition (2.4) and the estimate cited in the proof of Corollary 3.3 (also tacitly used in the proof of §3.4), it is clear that $u(R\tau) \cdot \overline{u_{,\rho}}(R\tau) = - i\bar{\kappa}|u(R\tau)|^2 + o(1/R^2)$ as $R \to \infty$, uniformly in τ for $|\tau| = 1$. Consequently,

$$Im \int_{\partial B_R} \zeta u \overline{u_{,\rho}} \, d\lambda_{\partial B_R} = - Re(\zeta \bar{\kappa}) \cdot \int_{\partial B_R} |u|^2 \, d\lambda_{\partial B_R} + o(1), \qquad \text{as } R \to \infty.$$

Using these facts and assuming that $Im < \zeta u|_\Gamma^2 , u_{,\nu}^2 > \geqslant 0$, we get

$$0 \leqslant Im < \zeta u|_\Gamma^2 , u_{,\nu}^2 > = - \{ - Im(\zeta \bar{\kappa}^2) \cdot \int_{\Omega_+(R)} |u|^2 \, d\lambda + Im\zeta \cdot \int_{\Omega_+(R)} |\text{grad } u|^2 \, d\lambda$$

$$+ Re(\zeta \bar{\kappa}) \cdot \int_{\partial B_R} |u|^2 d\lambda_{\partial B_R} \} + o(1),$$

holding for all sufficiently large $R > 0$. Then the limit, as $R \to \infty$, of the expression on the right exists and is nonnegative, whence the limit of the nonnegative expression in brackets must exist and equal zero (recall, from §2, that $Re(\zeta \bar{\kappa}) \geqslant 0$). In turn, the limit of each (nonnegative) term within the brackets exists and equals zero. Now, if $Im\kappa = 0$, then $Im\zeta = 0$ but $Re(\zeta \bar{\kappa}) = \zeta\kappa > 0$, and the vanishing of u follows from Rellich's Lemma (which remains valid in the present setting, since it can be proven without regard for the behavior near Γ; cf. [15] or [2, Lemma 3.11 or Lemma 3.14]); if $Im\kappa > 0$, then at least one of $Im\zeta$, $-Im(\zeta \bar{\kappa}^2)$ is positive, from which it is easy to see that $u = 0$ in either case. □.

As anticipated, we can now easily prove:

3.6 Corollary

For each $g \in H_0$, $ER(g|B;\kappa)$ can have at most one solution function, *i.e.*, $ER(0|B;\kappa)$ has only the trivial solution function.

Proof: Let $u \in W(\Omega_+;\kappa)$, with $u_{,\nu}^2 + Bu|_\Gamma^2 = 0$. Recalling $(2.2)_2$, $\zeta_\kappa^B \in Z_\kappa$ and

$$Im < \zeta_\kappa^B u|_\Gamma^2 , u_{,\nu}^2 > = -Im < \zeta_\kappa^B u|_\Gamma^2 , Bu|_\Gamma^2 > = Im < \overline{\zeta_\kappa^B} Bu|_\Gamma^2 , u|_\Gamma^2 > \geqslant 0,$$

so $u = 0$. □.

4. SINGLE- AND DOUBLE-LAYER POTENTIALS. INTEGRAL OPERATORS

We have collected here the results concerning single- and double-layer potentials for the Helmholtz operator and the allied operators in H_0 that shall be needed subsequently.

Recall the definition of the fundamental solution E and the associated notations that were established in §3. Whenever $\phi \in L_1(\Gamma)$ (in particular, for $\phi \in H_0$), the single-layer potential $V\{\phi\}$ and the double-layer potential $W\{\phi\}$ are the complex functions defined in $\Omega_- \cup \Omega_+$ by setting

$$\left. \begin{aligned} V\{\phi\}(x) &:= \int_\Gamma E_x \cdot \phi \, d\lambda_\Gamma , \\ W\{\phi\}(x) &:= \int_\Gamma E_{x,\nu} \cdot \phi \, d\lambda_\Gamma , \end{aligned} \right\} \quad x \in \Omega_- \cup \Omega_+ .$$

It is well known that $V\{\phi\}$ and $W\{\phi\}$ lie in $C^{\infty}(\Omega_- \cup \Omega_+)$, in fact are real-analytic, and are solutions of (2.3) in $\Omega_- \cup \Omega_+$. We write their restrictions to Ω_+ and Ω_- as, respectively,

$$V_{\pm}\{\phi\} := V\{\phi\} \,|\, \Omega_{\pm},$$
$$W_{\pm}\{\phi\} := W\{\phi\} \,|\, \Omega_{\pm}.$$

$V_+\{\phi\}$ and $W_+\{\phi\}$ satisfy the radiation condition (2.4).

Whenever $\phi \in H_o$, it can be shown that, for λ_Γ- almost every $x \in \Gamma$, $E_x \phi$ and $E_{,\nu(x)}(x;\cdot)\phi$ are in $L_1(\Gamma)$, and, moreover, that the functions $S\phi$ and $K\phi$ defined by

$$\left.\begin{aligned} S\phi(x) &:= \int_\Gamma E_x \cdot \phi \, d\lambda_\Gamma, \\ K\phi(x) &:= \int_\Gamma E_{,\nu(x)}(x;\cdot)\, \phi \, d\lambda_\Gamma, \end{aligned}\right\} \quad \lambda_\Gamma\text{-almost every } x \in \Gamma,$$

are in H_o. One can prove that the resultant linear operators $S, K: H_o \to H_o$ are compact. Their (Hilbert-space) adjoints are also integral operators and are given by, for $\phi \in H_o$,

$$\left.\begin{aligned} S^*\phi(x) &= \int_\Gamma \bar{E}_x \cdot \phi \, d\lambda_\Gamma, \\ K^*\phi(x) &= \int_\Gamma \bar{E}_{x,\nu} \cdot \phi \, d\lambda_\Gamma, \end{aligned}\right| \quad \lambda_\Gamma\text{-almost every } x \in \Gamma.$$

A presentation of these facts in the case $\kappa = 0$ can be found in Ref. 16, and the proofs given there can be carried over to the present situation in a simple manner. By $\bar{S}$, $\bar{K}$, and $\overline{K^*}$ we shall denote the integral operators on H_o with the respective conjugated kernels. Thus, $\bar{S} = S^*$ (since $E(\cdot;\cdot)$ is symmetric in its arguments), and it can be checked that $\overline{K^*} = (\bar{K})^*$. Observe that the latter operator is generated by the kernel $E_{x,\nu}(y) = E_{,\nu(x)}(x;y)$. We are using here the operator notation of Ref. 17.

We proceed to a listing of facts concerning these potentials and integral operators, providing for each result a reference to the literature, where a proof is to be found. Actually, Günter [18] concerns himself with the potential-theoretic case, corresponding to $\kappa = 0$, but his proofs can be easily modified to yield certain statements that we are about to make; it is with this understanding that we shall cite Ref. 18.

4.1 Essentially Bounded Density

Let $\phi \in L_\infty(\Gamma)$: Then $S\phi(x)$ exists for each $x \in \Gamma$, and $S\phi \in C_H(\Gamma)$; moreover, the definition $V\{\phi\}(x) := S\phi(x)$ for $x \in \Gamma$ produces an extension of $V\{\phi\}$ to all of $\mathbf{R}^3$ that is in $C_H(\mathbf{R}^3)$ [18]. Thus, $V_\pm\{\phi\} \in C_H(\Omega_\pm^-)$, with

$$V_\pm\{\phi\}|_\Gamma = S\phi. \tag{4.1}$$

S maps $L_\infty(\Gamma)$ into $C_H(\Gamma)$.

4.2 Continuous Density

(i) S, K, and $\overline{K^*}$ map $C(\Gamma)$ into $C_H(\Gamma)$; S and $\overline{K^*}$ map $C_H(\Gamma)$ into $C_H^1(\Gamma)$ (these follow from Ref. 2, Theorem 2.30 and p. 62).

Now, let $\phi \in C(\Gamma)$ in (ii) to (iv) (certainly, the assertions of §4.1 hold for ϕ in this case):

(ii) Each of $V_+\{\phi\}$ and $V_-\{\phi\}$ has a normal derivative on Γ in the uniform sense, given by, respectively,

$$V_\pm\{\phi\}_{,\nu}^{\zeta} := \lim_{\epsilon \to 0^+} \nu \bullet \{(\text{grad } V_\pm\{\phi\}) \; N_{\pm\epsilon}\} = \mp\phi + K\phi \quad \text{in } C(\Gamma) \tag{4.2}$$

([2, Theorem 2.19]; note that there a lower minus sign has been omitted in the definition of the normal derivatives).

(iii) $W_+\{\phi\} \in C(\bar{\Omega}_+)$ and $W_-\{\phi\} \in C(\bar{\Omega}_-)$, *i.e.*, each function can be continuously extended to the indicated closure, and the restrictions of the extensions to Γ are given by, respectively,

$$W_\pm\{\phi\}|_\Gamma = \mp\phi + \overline{K^*}\phi \tag{4.3}$$

[2, Theorem 2.13].

(iv) There holds

$$\lim_{\epsilon\to 0^+}\{\nu \bullet [(\operatorname{grad} W_+\{\phi\})\circ N_\epsilon] - \nu \bullet [(\operatorname{grad} W_-\{\phi\})\circ N_{-\epsilon}]\} = 0 \text{ in } C(\Gamma) \tag{4.4}$$

[2, Theorem 2.21]. As an immediate consequence, we see that if either of $W_+\{\phi\}_\nu^C$, $W_-\{\phi\}_\nu^C$ is known to exist, then the other must exist and there is equality between the two limits. We shall refer to this statement as the "uniform Lyapunov-Tauber Theorem."

Let us introduce the linear manifold $\mathbf{N}_\nu^C$ in H_0,

$$\mathbf{N}_\nu^C = \{\phi \in C(\Gamma) \mid \text{either } W_+\{\phi\}_\nu^C \text{ or } W_-\{\phi\}_\nu^C \text{ exists}\},$$

and the linear operator $W_\nu^C : \mathbf{N}_\nu^C \to H_0$ given by

$$W_\nu^C\phi := W_+\{\phi\}_\nu^C \; (= W_-\{\phi\}_\nu^C), \quad \phi \in \mathbf{N}_\nu^C,$$

always keeping in mind (iv) when working with W_ν^C. It will follow from the results of §5, Corollary 6.7 iv, and Corollary 6.12 that $\mathbf{N}_\nu^C$ is dense in H_0.

(v) If $\phi \in C_H(\Gamma)$, then $V_\pm\{\phi\} \in C_H^1(\bar{\Omega}_\pm)$, *i.e.*, the first partial derivatives of $V_\pm\{\phi\}$ in $\Omega_\pm$ possess continuous extensions to $\bar{\Omega}_\pm$, the extensions lying in $C_H(\bar{\Omega}_\pm)$ [2, Theorem 2.17], while $W_\pm\{\phi\} \in C_H(\bar{\Omega}_\pm)$ [2, Theorem 2.16].

(vi) If $\phi \in C_H^1(\Gamma)$, then $W_\pm\{\phi\} \in C_H^1(\bar{\Omega}_\pm)$ [2, Theorem 2.23]. In consequence, we have the inclusion $C_H^1(\Gamma) \subset \mathbf{N}_\nu^C$ (cf. Remark 2.2 c).

4.3 Density in $\mathbf{H}_0$

In the results that we are about to state, the assertions concerning convergence in H_0 are proven in Ref. 5, while those concerning convergence pointwise λ_Γ-a.e. on Γ can be verified by employing the reasoning in Ref. 18. Let $\phi \in H_0$.

(i) Each of $V_+\{\phi\}$ and $V_-\{\phi\}$ has a normal trace on Γ in the L_2-sense, given by

$$V_\pm\{\phi\}_\Gamma = \lim_{\epsilon\to 0^+} V_\pm\{\phi\}\circ N_{\pm\epsilon} = S\phi \quad \text{in } H_0; \tag{4.5}$$

the limiting relations indicated also hold λ_Γ-a.e. on Γ.

(ii) Each of $V_+\{\phi\}$ and $V_-\{\phi\}$ has a normal derivative on Γ in the L_2-sense, given by

$$V_\pm\{\phi\}_\nu = \lim_{\epsilon\to 0^+} \nu \bullet [(\operatorname{grad} V_\pm\{\phi\})\circ N_{\pm\epsilon}] = \pm\phi + K\phi \quad \text{in } H_0; \tag{4.6}$$

the indicated limiting relations also hold λ_Γ-a.e. on Γ.

(iii) Each of $W_+\{\phi\}$ and $W_-\{\phi\}$ has a normal trace on Γ in the L_2-sense, given by

$$W_\pm\{\phi\}|_\Gamma^2 := \lim_{\epsilon \to 0^+} W_\pm\{\phi\} \circ N_{\pm\epsilon} = \mp\phi + \overline{K^*}\phi \quad \text{in } H_0; \tag{4.7}$$

the limiting relations indicated also hold λ_1-a.e. on Γ.

(iv) There holds

$$\lim_{\epsilon \to 0^+}\{\nu \bullet [(\operatorname{grad} W_+\{\phi\}) \circ N_\epsilon] - \nu \bullet [(\operatorname{grad} W_-\{\phi\}) \circ N_{-\epsilon}]\} = 0 \text{ in } H_0. \tag{4.8}$$

Consequently, if either of $W_+\{\phi\}_\nu^2$, $W_-\{\phi\}_\nu^2$ is known to exist, then the other must exist and there is equality between the two limits. We shall refer to this statement as the "L_2-Lyapunov-Tauber Theorem."

We introduce the linear manifold $\mathbf{N}_\nu^2$ in H_0 as

$$\mathbf{N}_\nu^2 := \{\phi \in H_0 \mid \text{either } W_+\{\phi\}_\nu^2 \text{ or } W_-\{\phi\}_\nu^2 \text{ exists}\},$$

and the linear operator $W_\nu^2 : \mathbf{N}_\nu^2 \to H_0$ given by

$$W_\nu^2\phi := W_+\{\phi\}_\nu^2 (= W_-\{\phi\}_\nu^2), \quad \phi \in \mathbf{N}_\nu^2,$$

remaining mindful of (iv) when working with W_ν^2. On the basis of the remark made in §4.2 and the obvious inclusion $\mathbf{N}_\nu^c \subset \mathbf{N}_\nu^2$, we anticipate discovering that $\mathbf{N}_\nu^2$ is dense in H_0; the latter result will also follow from Corollary 6.7 iii and Corollary 6.11, in a more direct manner. An alternate characterization of $\mathbf{N}_\nu^2$ is to be found in Corollary 8.3. Of course, W_ν^2 is an extension of W_ν^c.

It is important to note the inclusions $V_\pm\{\phi\} \in W(\Omega_\pm;\kappa)$ for $\phi \in H_0$ and $W_\pm\{\phi\} \in W(\Omega_\pm;\kappa)$ for $\phi \in \mathbf{N}_\nu^2$, which follow from (i), (ii) and (iii), (iv), respectively.

The following statement is of the form of an *a priori* regularity theorem for solutions of certain Fredholm integral equations of the second kind in H_0, for which we will find a use on several occasions.

4.4 Lemma

Let $\tilde{K}$ denote one of the operators $\pm K$, $\pm K^*$, $\pm \overline{K}$, or $\pm \overline{K^*}$ on H_0. If $\phi \in H_0$ is such that $\phi + \tilde{K}\phi \in C(\Gamma)$, then the inclusion $\phi \in C(\Gamma)$ must also hold.

Proof: It is easy to see that $\tilde{K}$ is an operator with a weakly singular kernel; in fact, its kernel is of the form $\tilde{K}_0(x,y)/|x-y|^{2-\beta}$, with $0 < \beta < 1$ and $\tilde{K}_0$ continuous on $\Gamma \times \Gamma$. In Ref. 16, Theorem 8.6.1, Mikhlin proves that the inclusions $\hat{\phi} \in L_2(U)$ and $(I+\hat{K})\hat{\phi} \in C(\overline{U})$ imply $\hat{\phi} \in C(\overline{U})$ when U is a bounded open set in $\mathbb{R}^n$ and $\hat{K}$ is an integral operator on $L_2(U)$ with weakly singular kernel of the form $\hat{K}_0(x,y)/|x-y|^\alpha$, wherein $0 \leqslant \alpha < n$ and $\hat{K}_0$ is continuous on $\overline{U} \times \overline{U}$. He implies [16, p. 372] that the same proof, *mutatis mutandis*, serves also to substantiate the present Lemma, in which the underlying measure space is the two-dimensional manifold Γ of class C^2, with the measure λ_1. One can check that this is indeed so. We shall let this suffice for the proof. □

5. A FAMILY COMPLETE IN H_0

We shall find it essential, from the standpoint of the existence result which is our first aim, to know of the existence of a set that is complete in H_0 and intimately related to the operator $\Delta + \kappa^2$, in a sense to be specified. Moreover, it will turn out that any particular family of this sort can serve in the construction of solutions. Accordingly, here we shall set down the properties required of such a family, subsequently remarking on the well-established existence of collections of this sort.

5.1 Notation

Throughout, $\{v_n\}_{n=1}^{\infty}$ shall denote a (countable) family in $W(\Omega_+,\kappa)$ such that $\{v_{n,\nu}|_\Gamma\}_{n=1}^{\infty}$ is complete in H_0 and (for convenience) linearly independent. Then the Gram-Schmidt orthonormalization procedure (cf., *e.g.*, [19]), when applied to $\{v_{n,\nu}|_\Gamma\}_{n=1}^{\infty}$ generates an orthonormal basis for H_0. Thus, there exists a collection $\{a_{nj} \mid j = 1,\ldots,n, n = 1,2,\ldots\} \subset \mathbb{C}$ such that for the family $\{\hat{v}_n\}_{n=1}^{\infty} \subset W(\Omega_+,\kappa)$ defined by

$$\hat{v}_n := \sum_{j=1}^{n} a_{nj} v_j, \qquad n = 1,2,\ldots, \tag{5.1}$$

the corresponding collection $\{\hat{v}_{n,\nu}|_\Gamma\}_{n=1}^{\infty}$ of Neumann data is orthonormal and complete in H_0.

5.2 Remarks

(a) Since the terminology is not standardized in the literature, we shall state explicitly that the completeness hypothesis for $\{v_{n,\nu}|_\Gamma\}_{n=1}^{\infty}$ requires that 0 be the single element of H_0 that is orthogonal to $\{v_{n,\nu}|_\Gamma\}_{n=1}^{\infty}$. Of course, an equivalent condition is that $\{v_{n,\nu}|_\Gamma\}_{n=1}^{\infty}$ be "fundamental," or "closed," in H_0, *i.e.*, that the collection of all linear combinations of finite numbers of elements selected from the family be dense in H_0.

(b) We show in §6 that the families $\{v_n|_\Gamma\}_{n=1}^{\infty}$ and $\{v_{n,\nu}|_\Gamma + Bv_n|_\Gamma\}_{n=1}^{\infty}$ are also complete in H_0. Note that, *e.g.*, $\{\hat{v}_n|_\Gamma\}_{n=1}^{\infty}$ will not in general be orthogonal in H_0.

5.3 Examples

Various investigators have identified specific families possessing the properties required in §5.1, *e.g.*, Müller and Kersten [20], and Limić [21].

(a) References 20 and 21 give conditions on a sequence $(x_n)_1^{\infty}$ of points in Ω_- that are sufficient to ensure the completeness of $\{E_{x_n,\nu}\}_{n=1}^{\infty}$ in H_0.

(b) Let Ω_- be connected. Let 0 lie in Ω_- and coincide with the origin of a spherical coordinate system. Define the family $\{V_{lm} \mid m = -l,\ldots,l, l = 0,1,2,\ldots\}$ of "outgoing spherical κ-wave functions" in $\mathbf{R}^3\setminus\{0\}$ according to

$$V_{lm}(x) := h_l^{(1)}(\kappa|x|)\cdot P_l^{|m|}(\cos\theta_x)\cdot\begin{cases}\sin|m|\phi_x, & m < 0\\ \cos m\phi_x, & m \geq 0\end{cases}, \tag{5.2}$$

in which $(|x|,\theta_x,\phi_x)$ are the spherical coordinates of $x \neq 0$, $h_l^{(1)}$ is the spherical Hankel function of the first kind and order l, and P_l^k is the associated Legendre function ("on the cut") of order k and degree l [22]. By selecting any convenient bijection $(l,m)\rightarrow n$ of $\{(l,m) \mid m = -l,\ldots,l, l = 0,1,2,\ldots\}$ onto the set of positive integers, we obtain a family $\{V_n\}_{n=1}^{\infty} \subset W(\Omega_+,\kappa)$. In Ref. 20, it is shown that $\{V_{n,\nu}|_\Gamma\}_{n=1}^{\infty}$ is complete in H_0, while the linear independence of this set is easily verified (*e.g.*, as in Ref. 4). Thus, $\{V_n\}_{n=1}^{\infty}$ fulfills the conditions placed upon $\{v_n\}_{n=1}^{\infty}$ in §4.1, when Ω_- is connected. It should now be clear how to construct an example of a family as in §4.1 when Ω_- is not connected, by considering the components of Ω_- (which must be at most countably infinite in number).

With the results of §3 and §4 at our disposal, we can sketch a proof of the completeness property cited in §5.3(b) which differs from that given in Ref. 20.

5.4 Proposition

With the notation established in §5.3(b), if Ω_- is connected, then $\{V_{lm,\nu}|_\Gamma \mid m = -l,\ldots,l, l = 0, 1, 2,\ldots\}$ is complete in H_0.

Proof. In Ref. 23, the facts presented here in §4.2 and the uniqueness theorem for the classical exterior Neumann problem are used to show that if $f \in C(\Gamma)$ and

$$< f, \bar{Y}_{m,l} > \; = \; 0, \; m = -l, \ldots, l, \; l = 0, 1, 2, \ldots, \tag{5.3}$$

then $f = 0$. We show here that the same type of argument, using instead the results in §4.3 and Corollary 3.6, serves to secure the present stronger statement.

Let Ω_- be connected. Suppose that $f \in H_-$ and (5.3) holds. Then, just as in Ref. 23, we may use the well-known expansion

$$E(x, y) = \sum_{l}^{\cdot} \sum_{m} \gamma_{lm} \; RegY_{lm}(x) \; Y_{lm}(y), \quad x < y,$$

wherein $RegY_{lm}$ is obtained by replacing $h^{(1)}$ in (5.2) with the spherical Bessel function j, and the γ_{lm} are certain complex numbers, and convergence properties of this series to deduce that $\mathcal{B}_-\{f\}$ must vanish in a ball contained in Ω_-. Thus, $\mathcal{B}_-\{f\} = 0$, since Ω_- is connected and $\mathcal{B}_-\{f\}$ is real-analytic in Ω_-. Clearly, $\mathcal{B}_-\{f\}_-^{\partial} = 0$, so §4.3 (iv) says that $\mathcal{B}_+\{f\}_+^{\partial} = 0$. With §4.3 (iii), we see that $\mathcal{B}_+\{f\} \in \mathcal{B}(\Omega_+, \kappa)$ is a solution function for $ER(0, 0, \kappa)$, and so $\mathcal{B}_+\{f\} = 0$, by Corollary 3.6. Using (4.7), we now get

$$f = \frac{1}{2}\{\mathcal{B}_-\{f\}_-^{\partial} - \mathcal{B}_+\{f\}_+^{\partial}\} = 0.$$

Consequently, $\{\bar{Y}_{m,l}\}$, and then also $\{Y_{m,l}\}$, is complete in H_-.

Similar reasoning allows one to show that $\{Y_{m,l}\}$ is complete in H_+ as well. As already noted, we shall later show how completeness of the latter set follows directly from that of $\{Y_{m,l}\}$, so we shall not pursue the matter further here.

6. THE BOUNDARY-DATA OPERATOR

So far as we know at this point, the set $\mathbf{M}_+^{\partial} = \{u_+^{\partial} : u \in \mathcal{B}(\Omega_+, \kappa)\}$ is merely a linear manifold in H_+; since the complete set $\{v_n{}_+^{\partial}\}_n$ lies in $\mathbf{M}_+$, the latter is dense in H_+, however. One of our objectives is to show that, in fact, $\mathbf{M}_+^{\partial} = H_+$, which is equivalent to settling the existence question for the Neumann problem. Our first result says that the linear map $u_+^{\partial} \to u_+^{\partial}$ on $\mathbf{M}_+^{\partial}$ into H_- is at least a restriction of a compact operator on H_+. In the proof of this fact, we make decisive use of the existence result for the solution of the exterior Neumann problem in the weakly classical setting, for continuous Neumann data.

6.1. Lemma

There exists a unique compact linear operator $A: H_+ \to H_-$ such that

$$Au_+^{\partial} = u_+ \quad \text{for each } u \in \mathcal{B}(\Omega_+, \kappa). \tag{6.1}$$

Proof. There can exist at most one bounded $A: H_+ \to H_-$ with property (6.1), since, as pointed out, the set $\mathbf{M}_+^{\partial}$ defined above is dense in H_+.

Now let u denote an element of $\mathcal{B}(\Omega_+, \kappa)$. According to (3.11) in Corollary 3.4, $u = \frac{1}{2}\{\mathcal{V}_+\{u_+\} - \mathcal{W}_+\{u_+^{\partial}\}\}$, whence (4.5) and (4.7) imply that $u_+ = \frac{1}{2}\{Su_+ + u_+ - K^* u_+^{\partial}\}$, or

$$(I + \tilde{K}^*) u_+^{\partial} = Su_+^{\partial}, \quad u \in \mathcal{B}(\Omega_+, \kappa). \tag{6.2}$$

We consider separately the two possible cases, based upon $\mathcal{N}(I + K^*)$.

(a) Suppose that $\mathcal{N}(I + \bar{K}^*) = \{0\}$ the compactness of $\bar{K}^*$ then implies that $(I + \bar{K}^*)^{-1}$ is defined on H_γ and bounded. With the definition $A = (I + \bar{K}^*)^{-1}S$, we obtain a compact operator for which (6.1) follows from (6.2).

(b) Suppose that $\mathcal{N}(I + \bar{K}^*) \neq \{0\}$, we remark that this obtains iff κ^2 is an eigenvalue for the Dirichlet problem for $-\Delta$ in Ω_ι [17]. Let P denote the operator of orthogonal projection upon $\mathcal{N}(I + \bar{K}^*)$. To achieve our purpose, we shall produce compact A_1 and A_2 on H_γ such that

$$Pu|_{\Gamma}^{\partial} = A_1 u|_{\Gamma} \tag{6.3}$$

and for each $u \in \mathcal{W}(\Omega_\epsilon, \kappa)$

$$(I - P)u|_{\Gamma}^{\partial} = A_2 u|_{\Gamma} \tag{6.4}$$

Then, in $A = A_1 + A_2$ we shall have a compact operator satisfying (6.1), and the proof shall be complete.

Again appealing to the Fredholm Theory, we know that $\mathcal{N}(I + \bar{K}^*)$ and $\mathcal{N}(I + K)$ have the same finite dimension $d \geq 1$; let $\{\phi_k^*\}_{k=1}^d$ and $\{\phi_k\}_{k=1}^d$ be orthonormal bases for these subspaces, respectively. Upon recalling Lemma 4.4, it is evident that each of these bases is a subset of $C(\Gamma)$.

(b.1) Existence of A_1. From the observation just made, for each $k \in \{1, \ldots, d\}$ we know that there exists Φ_k^*, the (unique) weakly classical solution of $ER(\bar{\phi}_k^*, 0, \kappa)$, the Neumann problem with boundary data $\bar{\phi}_k^*$. $\Phi_k^* \in C^2(\Omega_\epsilon) \cap C(\bar{\Omega}_\epsilon)$, satisfies (2.3) in Ω_ϵ, and (2.4), while $\Phi_k^*|_{\Gamma}^{\partial}$ exists and equals $\bar{\phi}_k^*$ (cf., e.g., [2, Theorem 3.25]). In particular, $\Phi_k^* \in \mathcal{W}(\Omega_\epsilon, \kappa)$ for each k, so Corollary 3.3 can be applied to write

$$\int_\Gamma \{u|_{\Gamma}^{\partial}\, \bar{\Phi}_k^* - u|_{\Gamma}\, \Psi_k^*|_{\Gamma}\}\, d\lambda_1 = 0,$$

or

$$\langle u|_{\Gamma}^{\partial}, \phi_k^* \rangle = \langle u|_{\Gamma}, \bar{\Phi}_k^*|_{\Gamma} \rangle, \quad k = 1, \ldots, d,\ u \in \mathcal{W}(\Omega_\epsilon, \kappa).$$

With the latter equalities, our objective is quickly realized, for now

$$\begin{aligned} Pu|_{\Gamma}^{\partial} &= \sum_{k=1}^{d} \langle u|_{\Gamma}^{\partial}, \phi_k^* \rangle\, \phi_k^* \\ &= \sum_{k=1}^{d} \langle u|_{\Gamma}, \bar{\Phi}_k^*|_{\Gamma} \rangle\, \phi_k^*, \quad u \in \mathcal{W}(\Omega_\epsilon, \kappa), \end{aligned}$$

so the definition

$$A_1 f = \sum_{k=1}^{d} \langle f, \bar{\Phi}_k^*|_{\Gamma} \rangle\, \phi_k^*, \quad f \in H_\gamma,$$

produces a finite-rank, and therefore compact, operator on H_γ with (6.3) holding.

(b.2) Existence of A_2. Here, we employ a strategy applied by K. E. Atkinson [24]. Define the finite-rank operator L on H by

$$Lf = \sum_{k=1}^{d} \langle f, \phi_k^* \rangle \phi_k, \quad f \in H \tag{6.5}$$

Let us first show that $I + \bar{K}^* + L$ is injective: if $(I + \bar{K}^* + L)f = 0$, then $(I + \bar{K}^*)f = -Lf \in \mathcal{N}(I + \bar{K}) = \mathcal{R}(I + \bar{K}^*)^{\perp}$, so $(I + \bar{K}^*)f = 0$. Thus, we have both $f \in \mathcal{N}(I + \bar{K}^*)$ and $f \in \mathcal{N}(L)$, inclusions which imply, respectively,

$$f = \sum_{k=1}^{d} \langle f, \phi_k^* \rangle \phi_k^*$$

and

$$< f, \phi_k^* > = 0, \quad k = 1, \ldots, d.$$

Thus, $f = 0$, *i.e.*, $I + \bar{K}^* + L$ is injective. Since $\bar{K}^* + L$ is compact, $(I + \bar{K}^* + L)^{-1}$ is defined on H_o and bounded. Now, since $I - P$ is the (self-adjoint) orthogonal projection operator upon $\mathcal{N}(I + \bar{K}^*)^{\perp}$, directly from (6.5) we get

$$L(I - P) f = 0, \quad f \in H_o,$$

while (6.2) gives

$$(I + \bar{K}^*)(I - P) u|_\Gamma^2 = (I + \bar{K}^*) u|_\Gamma^2 = Su_{,\nu}^2, \quad u \in W(\Omega_{,};\kappa),$$

whence

$$(I + \bar{K}^* + L)(I - P) u|_\Gamma^2 = Su_{,\nu}^2, \quad u \in W(\Omega_{,};\kappa).$$

Therefore, taking $A_2 := (I + \bar{K}^* + L)^{-1}S$, (6.4) follows, and A_2 is clearly compact. □

We shall reserve the symbol A to denote the operator constructed in the preceding lemma. Our study of the generalized Robin problem is based upon the properties of this compact operator; the remainder of this section is devoted to the development of a number of these properties.

6.2 Corollary

The following operator relations hold:

$$\text{(i) } A(I + K) = S, \tag{6.6}$$

$$\text{(ii) } AW_\nu^2 = (-I + \bar{K}^*) N_\nu^2 \tag{6.7}$$

Proof. Let $f \in H_o$; by the results cited in §4, $V_{,}\{f\}$ is an element of $W(\Omega_{,};\kappa)$, with $V_{,}\{f\}|_\Gamma^2 = Sf$ and $V_{,}\{f\}_{,\nu}^2 = (I + K)f$, so, by (6.1), $A(I + K)f = Sf$. This proves (6.6).

Equality (6.7) follows by similar reasoning: if $f \in N_\nu^2$, then $W_{,}\{f\} \in W(\Omega_{,};\kappa)$, $W_{,}\{f\}|_\Gamma^2 = (-I + \bar{K}^*)f$, and $W_\nu^2 f = W_{,}\{f\}_{,\nu}^2$. Applying (6.1), we infer that (6.7) is correct.

We remark that (6.7) just says that the densely defined operator AW_ν^2 on N_ν^2 is bounded; its unique continuous extension to all of H is $-I + \bar{K}^*$.

Representations of A are available, in terms of the particular orthonormal basis $\{\tilde{v}_{n,\nu}^2\}_{n=1}^\infty$ constructed from the family $\{v_n\}_{n=1}^\infty \subset W(\Omega_{,};\kappa)$ in §5.

6.3 Lemma

A and its adjoint A^* have the pairs of companion representations

$$A = \sum_{n}^{\cdot} < \cdot, \tilde{v}_{n,\nu}^2 > \tilde{v}_n|_\Gamma, \tag{6.8}$$

$$A^* = \sum_{n=1}^{\cdot} < \cdot, \tilde{v}_n|_\Gamma^2 > \tilde{v}_{n,\nu}^2 \tag{6.9}$$

$$A = \sum_{n=1}^{\cdot} < \cdot, \tilde{v}_n|_\Gamma^2 > \tilde{v}_{n,\nu}^2, \tag{6.10}$$

$$A^* = \sum_{n=1}^{\infty} < \;, \check{v}_{n,\nu}^{2} > \check{v}_n|_{\Gamma}^{2}, \tag{6.11}$$

in the sense of convergence in the strong operator topology.

Proof: Recall the family $\{\hat{v}_n\}_{n=1}^{\infty} \subset W(\Omega_+;\kappa)$, generated in §5 from $\{v_n\}_{n=1}^{\infty} \subset W(\Omega_+,\kappa)$, *via* the Gram-Schmidt process in such a way that $\{\hat{v}_{n,\nu}^{2}\}_{n=1}^{\infty}$ is an orthonormal basis for H_o. Thus, whenever $f \in H_o$, since A is bounded, from (6.1) we find

$$\begin{aligned} Af &= A\sum_{n=1}^{\infty} < f,\hat{v}_{n,\nu}^{2} > \hat{v}_{n,\nu}^{2} = \sum_{n=1}^{\infty} < f,\hat{v}_{n,\nu}^{2} > A\,\hat{v}_{n,\nu}^{2} \\ &= \sum_{n=1}^{\infty} < f,\hat{v}_{n,\nu}^{2} > \hat{v}_n|_{\Gamma}^{2}, \end{aligned}$$

while

$$A^*f = \sum_{n=1}^{\infty} < A^*f,\hat{v}_{n,\nu}^{2} > \hat{v}_{n,\nu}^{2} = \sum_{n=1}^{\infty} < f,\hat{v}_n|_{\Gamma}^{2} > \hat{v}_{n,\nu}^{2},$$

proving (6.8) and (6.9).

Observe next that

$$< \hat{v}_{n,\nu}^{2}, \check{v}_k|_{\Gamma}^{2} > = < \hat{v}_n|_{\Gamma}^{2}, \check{v}_{k,\nu}^{2} >, \quad k,n = 1,2,\ldots,$$

by Corollary 3.3. Consequently, use of (6.9) shows that

$$\begin{aligned} A^*\check{v}_{k,\nu}^{2} &= \sum_{n=1}^{\infty} < \check{v}_{k,\nu}^{2}, \hat{v}_n|_{\Gamma}^{2} > \hat{v}_{n,\nu}^{2} = \sum_{n=1}^{\infty} < \check{v}_k|_{\Gamma}^{2}, \hat{v}_{n,\nu}^{2} > \hat{v}_{n,\nu}^{2} \\ &= \check{v}_k|_{\Gamma}^{2}, \quad k = 1,2,\ldots. \end{aligned} \tag{6.12}$$

Since $\{\check{v}_{n,\nu}^{2}\}_{n=1}^{\infty}$ is also an orthonormal basis for H_o, it is now easy to see how the validity of (6.11) follows from (6.12) and the boundedness of A^*, by reasoning as in the proof of (6.8), while (6.10) results from a computation similar to that which gave (6.9).

Perhaps it is of interest to isolate the facts underlying the strong convergence of the series in (6.8) to (6.11); observe that the sequences of terms in (6.8) and (6.11) are not generally orthogonal. We omit the easy proof of the following corollary of the proof of Lemma 6.3:

6.4 Corollary

The sequence $(< f,\hat{v}_n|_{\Gamma}^{2} >)_{n=1}^{\infty}$ lies in l_2 for each $f \in H_o$. The series $\sum_{n=1}^{\infty} \xi_n \hat{v}_n|_{\Gamma}^{2}$ converges (strongly) in H whenever $(\xi_n)_{n=1}^{\infty} \in l_2$.

Returning to the main line of development, we point out the simple connection between A and A^*.

6.5 Corollary

$$A^*g = \bar{A}g \quad \text{for each } g \in H. \tag{6.13}$$

Proof. This is an immediate consequence of (6.8) and (6.11).

The preceding statement, an extension of (6.12), says that A^* performs the Neumann-to-Dirichlet-data mapping for the collection of conjugates of elements of $\mathcal{W}(\Omega_-;\kappa)$, which are solutions for the operator $\Delta + \kappa^2$.

The following fact and its consequences are fundamental for our later reasoning.

6.6 Lemma

$$\mathcal{W}\{Af\} = \mathcal{V}\{f\} \quad \text{for each } f \in H_o . \tag{6.14}$$

Proof. Choose any $x \in \Omega_-$; then $E_x|\Omega_+ \in \mathcal{W}(\Omega_+;\kappa)$, so Corollary 3.3 yields, for any $f \in H_o$,

$$\int_\Gamma \{E_x \sum_{n=1}^{N} < f,\hat{v}_{n,i}^2 > \hat{v}_{n,i}^2 - E_{x,\nu} \sum_{n=1}^{N} < f,\hat{v}_{n,i}^2 > \hat{v}_n|_\Gamma^2 \} \, d\lambda_1 = 0 , \quad \text{for } N = 1,2,\ldots .$$

Since the partial sums appearing here converge in H_o to f and Af, respectively, we can let $N \to \infty$ to conclude that

$$\int_\Gamma \{E_x \cdot f - E_{x,\nu} \cdot Af\} \, d\lambda_1 = 0,$$

i.e.,

$$\mathcal{V}\{f\}(x) = \mathcal{W}\{Af\}(x) . \qquad \square$$

6.7 Corollary

The following operator relations hold.

(i) $(I + \bar{K}^*)A = S$, (6.15)

(ii) $AK = \bar{K}^*A$, (6.16)

(iii) $\mathscr{R}(A) \subset \mathrm{N}_i^2$ and $\mathcal{W}_i^2 A = -I + K$, (6.17)

(iv) $A(C(\Gamma)) \subset \mathrm{N}_i^c$ and $\mathcal{W}_i^c A = -I + K$ on $C(\Gamma)$. (6.18)

Proof. Let $f \in H_o$. From (6.14), we get $\mathcal{W}\{Af\}|_i^2 = \mathcal{V}\{f\}|_i^2$, or $(I+\bar{K}^*)Af = Sf$, proving (i). Now (ii) follows from (i) and (6.6). Further, (6.14) also implies that $\mathcal{W}_i^2 Af = \mathcal{W}\{Af\}_{,i}^2$ exists and equals $\mathcal{V}\{f\}_{,i}^2 = (-I + K)f$, which is the content of statement (iii). Similarly, (iv) also follows from (6.14). $\square$

In Corollary 8.3, *infra*, we shall show that $\mathscr{R}(A) = \mathrm{N}_i^2$. In any case, the inclusions of §6.7 iii, iv express a certain "smoothing" property of A; here are others, implied by §6.7 i.

6.8 Corollary

A maps $L_\infty(\Gamma)$ into $C_H(\Gamma)$ and $C_H(\Gamma)$ into $C_H^1(\Gamma)$.

Proof. In addition to §6.7 i, we shall use §4.1, §4.2 i, and Lemma 4.4. Let $f \in L_\infty(\Gamma)$; then $Sf \in C_H(\Gamma)$, so the equality $(I + \bar{K}^*)Af = Sf$ implies first that $Af \in C(\Gamma)$; therefore, $\bar{K}^*Af \in C_H(\Gamma)$, so $Af = Sf - \bar{K}^*Af \in C_H(\Gamma)$, verifying the first statement. In addition, we discover that $\bar{K}^*Af \in C_H^1(\Gamma)$ if $f \in L_\infty(\Gamma)$. Thus, when $f \in C_H(\Gamma)$, the inclusion $Sf \in C_H^1(\Gamma)$ follows and shows that $Af \in C_H^1(\Gamma)$, as well. $\square$

With the information concerning A that has been gathered to this point, we can examine the general form of elements of $\mathcal{W}(\Omega_-;\kappa)$.

6.9 Theorem

(i) Let $f \in H_0$. Define Φ_f in Ω_+ by

$$\Phi_f := \frac{1}{2}\{V_+\{f\} - W_+\{Af\}\}. \tag{6.19}$$

Then

(a) $\Phi_f \in W(\Omega_+;\kappa)$, with

and

$$\Phi_f|_\Gamma^2 := \lim_{\epsilon\to 0^+} \Phi_f \circ N_\epsilon = Af \qquad \text{in } H_o \tag{6.20}$$

$$\Phi_{f,\nu}^{\,2} := \lim_{\epsilon\to 0^+} \nu \bullet \{(\operatorname{grad}\Phi_f) \circ N_\epsilon\} = f \quad \text{in } H_o, \tag{6.21}$$

the convergence in (6.20) also holding pointwise λ_Γ-*a.e.* on Γ;

(b) if $f \in L_\infty(\Gamma)$, so that $Af \in C_H(\Gamma)$, then $\Phi_f \in C_H(\Omega_+)$, and the convergence in (6.20) is uniform on Γ, *i.e.*,

$$\Phi_f|_\Gamma = Af; \tag{6.22}$$

(c) if $f \in C(\Gamma)$, then the convergence in (6.21) is uniform on Γ, *i.e.*, $\Phi_{f,\nu}^{\,c}$ exists, and

$$\Phi_{f,\nu}^{\,c} = f; \tag{6.23}$$

(d) if $f \in C_H(\Gamma)$, then $\Phi_f \in C_H^1(\Omega_+^-)$.

(ii) If $u \in W(\Omega_+;\kappa)$, then $u = \Phi_{u_\nu^2}$.

Proof: (i.*a*) The properties of single- and double-layer potentials in Ω_+ and the form of definition (6.19) show that Φ_f is real-analytic and satisfies (2.3) in Ω_+, as well as the radiation condition (2.4). Moreover, using the results cited in §4.3, we compute, with (6.15),

$$\Phi_f|_\Gamma^2 = \frac{1}{2}\{Sf - (-I + \bar{K}^*)Af\} = \frac{1}{2}\{(I + \bar{K}^*)Af - (-I + \bar{K}^*)Af\}$$
$$= Af,$$

while from (6.17) we conclude that $\Phi_{f,\nu}^{\,2}$ exists, with

$$\Phi_{f,\nu}^{\,2} = \frac{1}{2}\{(I + K)f - W_\nu^2 Af\} = \frac{1}{2}\{(I + K)f - (-I + K)f\}$$
$$= f.$$

Thus, Φ_f is also L_2-regular at Γ, and (6.20) and (6.21) are correct. The pointwise-convergence assertion is clear.

(i.*b*) Now suppose that $f \in L_\infty(\Gamma)$: then $V_+\{f\} \in C_H(\Omega_+)$ (by the statements in §4.1) and the result §4.2.v says that also $W_+\{Af\} \in C_H(\Omega_+^-)$, since the inclusion $Af \in C_H(\Gamma)$ follows from Corollary 6.8. Thus, $\Phi_f \in C_H(\Omega_+^-)$. Use of (4.1) and (4.3) (instead of (4.5) and (4.7)) with (6.15) shows that the limit in (6.20) is uniform and, of course, (6.22) is true.

(i.*c*) Now suppose that $f \in C(\Gamma)$: then $V_+\{f\}_{,\nu}^{\,c} = (I + K)f$ by (4.2), and (6.18) tells us that $W_+\{Af\}_\nu^c$ exists and equals $(-I + K)f$. Statement (*c*) follows directly from these observations.

(i.d) Let $f \in C_H(\Gamma)$: then $V_+\{f\} \in C_H^1(\Omega_+)$ by §4.2.v, while $Af \in C_H^1(\Gamma)$ by Corollary 6.8, so §4.2.vi gives $W_+\{Af\} \in C_H^1(\Omega_+)$. The inclusion claimed in (d) clearly follows.

(ii) Let $u \in W(\Omega_+;\kappa)$: then $u|_\Gamma^+ = Au_{,\nu}^+$, so the equality $u = \Phi_{u_{,\nu}^+}$ is just a restatement of the conclusion of Corollary 3.4. $\Box$.

Theorem 6.9 implies that the mapping $u \mapsto u_{,\nu}^+$ carries $W(\Omega_+;\kappa)$ *onto* H_o (so that the map is in fact a bijection) and also shows how to represent each element of $W(\Omega_+;\kappa)$ in terms of its Neumann data alone. Already this theorem affords us the ability to deal with the Neumann problem $ER\,(g|0;\kappa)$, for any $g \in H_o$; the appropriate statement shall be subsumed as a special case of the results for the Robin problem, to be given in the next section. For the present, we are more interested in securing the following fact.

6.10 Corollary

Let $\zeta \in Z_\kappa$. Whenever $f \in H_o$ and $f \neq 0$,

$$Im < \zeta Af, f > \; < 0 . \tag{6.24}$$

Proof: Let f be a nonzero element of H_o. For the function $\Phi_f \in W(\Omega_+;\kappa)$ given by (6.19), we have $\Phi_f|_\Gamma^+ = Af$ and $\Phi_{f,\nu}^+ = f$. In particular, Φ_f cannot vanish in Ω_+, since $f \neq 0$. Therefore, using Corollary 3.5, we must have $Im < \zeta\, \Phi_f|_\Gamma^+, \Phi_{f,\nu}^+ > \; < 0$, which is just (6.24). $\Box$.

Note that (6.24) can be rewritten as $Re < -i\zeta Af, f > \; < 0$, so that the operator $-i\zeta A$ is dissipative if $\zeta \in Z_\kappa$.

6.11 Corollary

A and A^* are injective operators. In particular, $\mathscr{R}(A)$ and $\mathscr{R}(A^*)$ are dense in H_o.

Proof: The injective property of A follows immediately from the statement of Corollary 6.10, then that of A^* is a consequence of Corollary 6.5. The second statement is now clear, since $\mathscr{R}(A)^\perp = \mathscr{N}(A^*)$, $\mathscr{R}(A^*)^\perp = \mathscr{N}(A)$. $\Box$.

6.12 Corollary

For any family $\{v_n\}_{n=1}^\infty$ as in §5, *i.e.*, which lies in $W(\Omega_+;\kappa)$ and is such that $\{v_{n,\nu}^+\}_{n=1}^\infty$ is complete in H_o, the family $\{v_n|_\Gamma^+\}_{n=1}^\infty$ is also complete in H_o.

Proof: If $f \in H_o$ and $< f, v_n|_\Gamma^+ > \, = 0$ for each n, then $< A^*f, v_{n,\nu}^+ > \, = 0$ for each n, so $A^*f = 0$. Therefore, $f = 0$. $\Box$.

We have already observed that the dissipative nature of $i\overline{\zeta_\kappa^B}\, B$ implies that $ER\,(g|B;\kappa)$ can have at most one solution for any $g \in H_o$. In fact, the same hypothesis provides for the existence of a solution, as well; this is substantiated in the next section, as a consequence of the next result. The "strictly dissipative" property of $-i\zeta_\kappa^B A$ (Corollary 6.10) is also fundamental.

6.13 Corollary

The operators $I + BA$ and $I + B^*A^*$ are injective. Thus, each possesses a bounded inverse defined on H.

Proof: Suppose that $f \in H_o$ and $(I + BA)f = 0$ [resp., $(I + B^*A^*)f = 0$]. Then (2.2) gives

$$Im < \zeta_\kappa^B Af, f > \; = Im < Af, -\overline{\zeta_\kappa^B} BAf > \; = Im < \overline{\zeta_\kappa^B} BAf, Af > \; \geq 0$$

$$[\; Im < \zeta_\kappa^B Af, f > \; = Im < -\zeta_\kappa^B AB^*A^*f, f > \; = Im < \overline{\zeta_\kappa^B} BA^*f, A^*f > \; \geq 0] .$$

But Corollary 6.10 shows that such an inequality can hold only when $f = 0$. Thus, $I + BA$ $\{I + B^*A^*\}$ is injective. Since BA $\{B^*A^*\}$ is compact, the second part of the statement now follows from the Fredholm Theory. □

Immediately, we discover further classes of complete families in H_a:

6.14 Corollary

For any family $\{v_n\}_{n=1}^{\infty}$ as in §5, *i.e.*, which lies in $W(\Omega_+;\kappa)$ and is such that $\{v_{n,\nu}^{\,2}\}_{n=1}^{\infty}$ is complete in H_a, the collections $\{v_{n,\nu}^{\,2} + Bv_n|_\Gamma^2\}_{n=1}^{\infty}$ and $\{\bar{v}_{n,\nu}^{\,2} + B^*\bar{v}_n|_\Gamma^2\}_{n=1}^{\infty}$ are also complete in H_a.

Proof: For any $f \in H_a$ and $n = 1,2,\ldots$, we have

$$< f, v_{n,\nu}^{\,2} + B^*v_n|_\Gamma^2 > \; = \; < f, (I + BA)v_{n,\nu}^{\,2} > \; = \; < (I + A^*B^*)f, v_{n,\nu}^{\,2} > ,$$

$$< f, \bar{v}_{n,\nu}^{\,2} + B^*\bar{v}_n|_\Gamma^2 > \; = \; < f, (I + B^*A^*)\bar{v}_{n,\nu}^{\,2} > \; = \; < (I + AB)f, \bar{v}_{n,\nu}^{\,2} > .$$

The desired conclusions can be drawn from these equalities, since $I + A^*B^*$ and $I + AB$ are injective along with $I + BA$ and $I + B^*A^*$, respectively. □.

For the concise formulation of a continuous-dependence assertion to be given in §7, we shall introduce a topology for $W(\Omega_+;\kappa)$ that will make the latter into a locally convex linear topological space. For this purpose, we first note that, since $Im\,\kappa \geqslant 0$, there exist sets of positive numbers $\{m_n(\kappa)\}_{n=1}^{\infty}$ and $\{m_n'(\kappa)\}_{n=1}^{\infty}$ such that, whenever α is a 3-index,

$$|E_\alpha(x;y)| \leqslant m_{|\alpha|}(\kappa) \cdot \sum_{k=1}^{|\alpha|+1} \frac{1}{|y - x|^k}, \qquad x,y \in \mathbb{R}^3,\ x \neq y,$$

and

$$|E_{\nu(y),\alpha}(x;y)| \leqslant m'_{|\alpha|}(\kappa) \cdot \sum_{k=1}^{|\alpha|+2} \frac{1}{|y - x|^k}, \qquad x \in \mathbb{R}^3, y \in \Gamma, x \neq y;$$

one can easily check that this is so. Consequently, with the notation introduced in (6.19), we can readily conclude that there exist positive numbers $\{\tilde{M}_n(\kappa)\}_{n=1}^{\infty}$ such that, for each $f \in H_a$, each 3-index α, and each $x \in \Omega_+$, writing

$$d_\Gamma(x) := \text{dist}\,(x,\Gamma) := \inf_{y \in \Gamma} |y - x|,$$

we have

$$|\Phi_{f,\alpha}(x)| \leqslant \tilde{M}_{|\alpha|}(\kappa) \cdot \sum_{k=1}^{|\alpha|+2} \frac{1}{d_\Gamma^k(x)} \cdot \{\int_\Gamma |f|\, d\lambda_\Gamma + \int_\Gamma |Af|\, d\lambda_\Gamma\}$$

$$\leqslant \tilde{M}_{|\alpha|}(\kappa) \cdot \sum_{k=1}^{|\alpha|+2} \frac{1}{d_\Gamma^k(x)} \cdot \{\lambda_\Gamma(\Gamma)\}^{1/2} \cdot \{1 + ||A||\} \cdot ||f|| ; \tag{6.25}$$

in particular, then

$$\Phi_{f,\alpha}(\rho\tau) = O\left(\frac{1}{d_\Gamma(\rho\tau)}\right) = O\left(\frac{1}{\rho}\right), \text{ as } \rho \to \infty, \text{ uniformly in } \tau \text{ for } |\tau| = 1. \tag{6.26}$$

Thus, whenever $F \subset \Omega_+$ is closed in $\mathbb{R}^3$, each $\Phi_{f,\alpha}|F$ is bounded, $|\Phi_{f,\alpha}|F|$ taking on its *supremum*. Moreover, under the same hypothesis on F, (6.25) also shows that there are positive numbers $\{M_n(\kappa,F)\}_{n=1}^{\infty}$, depending on F only through dist(F,Γ), for which

$$\max_{x \in F} |\Phi_{f,\alpha}(x)| \leqslant M_{|\alpha|}(\kappa;F) \cdot \{1 + ||A||\} \cdot ||f||, \quad \text{for each } f \in H_a \text{ and 3-index } \alpha. \tag{6.27}$$

For $u \in W(\Omega_+,\kappa)$, we know that $u = \Phi_f$ if $f = u_\nu^{\,2}$, so (6.25) to (6.27) can be modified to yield corresponding statements concerning any such u.

Next, let $\{F_n\}_{n=1}^{\infty}$ be a (countable) collection of subsets of Ω_+ closed in $\mathbb{R}^3$ such that $F_1 \subset F_2^o \subset F_2 \subset \ldots \subset F_n \subset F_{n+1}^o \subset \ldots$, $\bigcup_{n=1}^{\infty} F_n = \Omega_+$, and F_1 contains the complement of some ball. According to what has been said, we can define a family $\{p_n\}_{n=1}^{\infty}$ of nonnegative functions on $W(\Omega_+;\kappa)$ by setting

$$p_n(u) := \max\{|u_{,\alpha}(x)| \mid x \in F_n, |\alpha| \leqslant n\}. \tag{6.28}$$

It is a simple matter to verify that each p_n is, in fact, a norm on $W(\Omega_+;\kappa)$ (so that the family is certainly separating). Thus, using a familiar construction (cf., *e.g.*, [25]), $\{p_n\}_{n=1}^{\infty}$ induces a locally convex topology on $W(\Omega_+;\kappa)$ with respect to which the linear space operations are continuous; since $p_n \leqslant p_m$ if $n < m$, a local base at 0 for this topology is given by the collection $\{U_n\}_{n=1}^{\infty}$, wherein

$$U_n := \left\{u \in W(\Omega_+;\kappa) \mid p_n(u) < \frac{1}{n}\right\}, \quad n = 1,2,\ldots. \tag{6.29}$$

The topology is metrizable, since the local base displayed is countable. Henceforth, all topological statements concerning $W(\Omega_+;\kappa)$ shall refer to the structure just introduced. For example, it is clear that a sequence $(u_n)_{n=1}^{\infty}$ in $W(\Omega_+;\kappa)$ converges to $u_0 \in W(\Omega_+;\kappa)$ *iff*, for each 3-index α, $(u_{n,\alpha})_{n=1}^{\infty}$ converges uniformly to $u_{0,\alpha}$ on each subset of Ω_+ closed in $\mathbb{R}^3$. The following simple result should be interpreted in the light of the latter statement.

6.15 Proposition

With the notation of (6.19), the mapping $f \to \Phi_f$ is continuous from H_0 onto $W(\Omega_+;\kappa)$.

Proof: It is easy to see that a linear operator $L : X \to W(\Omega_+;\kappa)$, wherein $(X, \|\cdot\|_X)$ is a complex normed linear space, is continuous *iff* there exist positive constants $\{\hat{M}_n\}_{n=1}^{\infty}$ such that $p_n(Lf) \leqslant \hat{M}_n \|f\|_X$ for each $f \in X$ and $n = 1,2,\ldots$. Now the conclusion of the proposition follows from (6.27), in view of the definition (6.28). □.

7. EXISTENCE, CONTINUOUS-DEPENDENCE, AND REGULARITY RESULTS

With the preparation provided in §6, the existence question for the generalized exterior Robin problem can be rapidly settled.

7.1 Theorem

For each $g \in H_0$, problem $ER(g|B;\kappa)$ is solved *via* the function $u_g := \Phi_{(I+BA)^{-1}g}$, *i.e.*, by $\frac{1}{2}\{V_+\{f_g\} - W_+\{Af_g\}\}$, in which f_g is the unique element of H_0 such that $(I+BA)f_g = g$.

Proof: Choose $g \in H_0$. Using Corollary 6.13, set $f_g := (I+BA)^{-1}g$, and construct Φ_{f_g} as in (6.19). By Theorem 6.9, $\Phi_{f_g} \in W(\Omega_+;\kappa)$, while

$$\Phi_{f_g,\nu}|_\Gamma^{\wedge} + B\Phi_{f_g}|_\Gamma^{\wedge} = (I+BA)f_g = g.$$

We already know that there can exist at most one such function in $W(\Omega_+;\kappa)$, so all requirements set down in §2.4 have been met. □.

Continuous dependence of the solution function jointly on the boundary-data function g and the operator B can be phrased and proven as follows:

7.2 Theorem

Recall the linear topological space structure with which $W(\Omega_+;\kappa)$ has been equipped in §6. Denote by $\mathscr{B}(\kappa)$ the metric space of all bounded linear $B : H_0 \to H_0$ for which $(2.2)_2$ holds, equipped with

the inherited operator-norm topology. For $g \in H_o$ and $B \in \mathscr{B}(\kappa)$, let u_g^B denote the solution function for $ER(g|B;\kappa)$. Let $F \subset \Omega_-$ be closed in $\mathbb{R}^3$. Then, with $\{M_n(\kappa;F)\}_{n=1}^{\infty}$ as in (6.27), we have the estimates

$$\max_{x \in F} | u_{g_2;\alpha}^{B_2}(x) - u_{g_1;\alpha}^{B_1}(x) | \leqslant M_{|\alpha|}(\kappa;F) \cdot \{1 + ||A||\} \cdot \{||(I + B_1A)^{-1}|| \cdot ||g_2 - g_1||$$

$$+ \frac{||A|| \cdot ||(I+B_1A)^{-1}||^2 \cdot ||B_2 - B_1||}{1 - ||A|| \cdot ||(I+B_1A)^{-1}|| \cdot ||B_2 - B_1||} \cdot ||g_2||\}, \tag{7.1}$$

holding whenever α is a 3-index, g_1 and $g_2 \in H_o$, and B_1 and $B_2 \in \mathscr{B}(\kappa)$ with $||B_2 - B_1|| < ||A||^{-1} \cdot ||(I + B_1A)^{-1}||^{-1}$. Thus, the mapping $(g,B) \mapsto u_g^B$ is continuous on $H_o \times \mathscr{B}(\kappa)$ into $W(\Omega_-;\kappa)$.

Proof: Let α be a 3-index, and choose $g_1, g_2 \in H_o$ and B_1, $B_2 \in \mathscr{B}(\kappa)$. Since $u_{g_k}^{B_k} = \Phi_{(I+B_kA)^{-1}g_k}$, $k = 1,2$, we obtain from (6.27)

$$\max_{x \in F} | u_{g_2;\alpha}^{B_2}(x) - u_{g_1;\alpha}^{B_1}(x) | \leqslant M_{|\alpha|}(\kappa;F) \cdot \{1 + ||A||\} \cdot \{||(I+B_2A)^{-1}g_2 - (I+B_1A)^{-1}g_1||\}. \tag{7.2}$$

Meanwhile,

$$||(I+B_2A)^{-1}g_2 - (I+B_1A)^{-1}g_1||$$

$$\leqslant ||(I+B_1A)^{-1}|| \cdot ||g_2 - g_1|| + ||(I+B_2A)^{-1} - (I+B_1A)^{-1}|| \cdot ||g_2||. \tag{7.3}$$

Now, since

$$(I+B_2A) = (I+B_1A) \cdot \{I - (I+B_1A)^{-1}(B_1 - B_2)A\},$$

when we suppose further that $||(I+B_1A)^{-1}|| \cdot ||B_1 - B_2|| \cdot ||A|| < 1$, we come easily to the estimate

$$||(I+B_2A)^{-1} - (I+B_1A)^{-1}|| \leqslant \sum_{n=0}^{\infty} \{||(I+B_1A)^{-1}|| \cdot ||B_1 - B_2|| \cdot ||A||\}^{n+1} \cdot ||(I+B_1A)^{-1}||$$

$$= \frac{||A|| \cdot ||(I+B_1A)^{-1}||^2 \cdot ||B_2 - B_1||}{1 - ||A|| \cdot ||(I+B_1A)^{-1}|| \cdot ||B_2 - B_1||}. \tag{7.4}$$

Combining (7.2), (7.3), and (7.4) produces (7.1), under the stated hypothesis on $||B_2 - B_1||$.

To verify the final assertion of the theorem, select $g_1 \in H_o$ and $B_1 \in \mathscr{B}(\kappa)$. Let $U \subset W(\Omega_-;\kappa)$ be a neighborhood of $u_{g_1}^{B_1}$ and choose N so large that $u_{g_1}^{B_1} + U_N \subset U$, wherein U_N is, of course, to be obtained from (6.29). In view of the definition (6.28), now (7.1) shows that we can find $\delta_\ell > 0$ so that $u_{g_2}^{B_2} - u_{g_1}^{B_1} \in U_N$ whenever $g_2 \in H_o$ and $B_2 \in \mathscr{B}(\kappa)$ with $||g_2 - g_1|| < \delta_\ell$, $||B_2 - B_1|| <$ min $\{\delta_\ell, ||A||^{-1} \cdot ||(I+B_1A)^{-1}||^{-1}\}$, whence $u_{g_2}^{B_2} \in U$ under the same restrictions. We conclude that the map $(g,B) \to u_g^B$ is continuous at (g_1,B_1). □

The regularity of the solution function for $ER(g|B;\kappa)$ depends upon the regularity of both g and B. We shall be content with a listing of the most obvious hierarchy of results in this direction, although it is likely that a more penetrating analysis would yield finer conclusions in the presence of some additional hypothesis on B (for example, if B were assumed to be an integral operator).

7.3 Theorem

For $g \in H_0$, u_g denotes the solution function for $ER(g|B;\kappa)$.

(i) For any $g \in H_0$, u_g takes on its Dirichlet data λ_Γ-a.e. in the sense of normal approach to Γ; more precisely,

$$u_g|_\Gamma^{2} = \lim_{\epsilon \to 0^+} u_g \circ N_\epsilon \quad \lambda_\Gamma\text{-}a.e. \text{ on } \Gamma;$$

(ii) if B maps H_0 into $L_\infty(\Gamma)$, then $u_g \in C_H(\Omega_+)$ whenever $g \in L_\infty(\Gamma)$;

(iii) if B maps H_0 into $C(\Gamma)$, then $u_{g,\nu}^{c}$ exists whenever $g \in C(\Gamma)$, whence u_g is weakly classical;

(iv) if B maps H_0 into $C_H(\Gamma)$, then $u_g \in C_H^1(\Omega_+)$ whenever $g \in C_H(\Gamma)$; in particular, u_g is a classical solution if B and g fulfill these conditions.

Proof: We have $u_g = \Phi_{f_g} = \frac{1}{2}\{V_+\{f_g\} - W_+\{Af_g\}\}$, in which $f_g = g - BAf_g$. Statement (i) follows immediately from the pointwise-convergence assertions in §4.3.i and §4.3.iii. If $Bf \in L_\infty(\Gamma)$ for each $f \in H_0$, and $g \in L_\infty(\Gamma)$, then also $f_g \in L_\infty(\Gamma)$, so $u_g \in C_H(\Omega_+)$ by §6.9.i.b, proving (ii). In a similar manner, (iii) and (iv) result from §6.9.i.c and §6.9.i.d, respectively. □.

8. CONSTRUCTION TECHNIQUES

We have seen that the construction of the solution function for $ER(g|B;\kappa)$ $(g \in H_0)$ can be effected by finding the unique $f_g \in H_0$ such that $(I+BA)f_g = g$, and then determining Af_g; in the case $B = 0$ (the Neumann problem), of course, $f_g = g$, so only the second task need be addressed. In any case, it is important to observe that it suffices to know Af_g alone, *i.e.*, that f_g need not be explicitly computed, for,

$$\Phi_{f_g}(x) = \frac{1}{2}\int_\Gamma \{ E_x \cdot f_g - E_{x,\nu} \cdot Af_g \} \, d\lambda_\Gamma$$

$$= \frac{1}{2}\int_\Gamma \{ E_x \cdot g - E_x \cdot BAf_g - E_{x,\nu} \cdot Af_g \} \, d\lambda_\Gamma$$

$$= \frac{1}{2}\int_\Gamma \{ E_x \cdot g - (\overline{B^* \overline{E}_x} + E_{x,\nu}) \cdot Af_g \} \, d\lambda_\Gamma, \qquad x \in \Omega_+ . \tag{8.1}$$

We shall discuss three approaches to the problem of computing Af_g: formulation of boundary-operator equations, orthonormalization, and formulation of generalized moment problems. We shall also show how Φ_{f_g} can be approximated by use of appropriate approximations for g.

Formulation of Boundary-Operator Equations

Here, the objective is the replacement of the original operator problem, involving $I + BA$, by one in which A does not appear (*i.e.*, in which only "computable" operators occur), which always possesses a unique solution, and the solution of which provides an algorithm for computing $Af_g (= A(I + BA)^{-1}g)$ whenever $g \in H_0$. By studying the operator relations available in §6, one discovers two semisystematic procedures for carrying out this program. Accordingly, we shall give two reformulations fulfilling the requirements imposed. In fact, the results are essentially extensions of boundary-integral reformulations already discovered for Neumann or Robin problems in various contexts.

The first reformulation, corresponding to the results of Refs. 3, 4, and 17, is described in the following statement:

8.1 Theorem

Corresponding to each $g \in H_o$, there exists a unique $\phi_g \in N_i^2$ such that

$$(I + \tilde{K}^* + SB)\phi_g = Sg \tag{8.2}$$

and

$$((I - K)B - W_i^2)\phi_g = (I - K)g; \tag{8.3}$$

this element is just $\phi_g = A(I + BA)^{-1}g$. If $I + \tilde{K}^*$ is injective [resp., if $I - \tilde{K}^*$ is injective] then, for each $g \in H_o$, (8.2) implies (8.3) [resp., (8.3) implies (8.2)], so ϕ_g is completely characterized by (8.2) [resp., by (8.3)] for that case.

Proof. Let $f, g \in H$. We set up a chain of equivalent statements, (i) through (v):

$$(i) \qquad (I+BA)f = g.$$

Since A is injective, (i) is true *iff*

$$(ii) \qquad (I+AB)Af = Ag.$$

In turn, (ii) is clearly equivalent to

$$(iii) \qquad \begin{cases} (I+\tilde{K}^*)(I+AB)Af = (I+\tilde{K}^*)Ag, \\ (I-\tilde{K}^*)(I+AB)Af = (I-\tilde{K}^*)Ag. \end{cases}$$

From (6.7), (6.15), (6.16), and the first assertion in (6.17), we find that (iii) is equivalent to

$$(iv) \qquad \begin{cases} (I+\tilde{K}^* + SB)Af = Sg, \\ -AW_i^2Af + A(I-K)BAf = A(I-K)g. \end{cases}$$

Finally, once more using the fact that A is injective, (iv) holds *iff*

$$(v) \qquad \begin{cases} (I+\tilde{K}^* + SB)Af = Sg, \\ ((I-K)B - W_i^2)Af = (I-K)g. \end{cases}$$

Now, (i) obtains *iff* $f = (I + AB)^{-1}g$, and, as already noted, $\mathcal{R}(A) \subset N_i^2$, whence the first statement of the theorem follows. Next, assume that $I+\tilde{K}^*$ [resp., $I-\tilde{K}^*$] is injective: then a string of equivalent statements results even when the second [resp., first] equality in each of (iii), (iv), and (v) is deleted, while (i) still implies the second [resp., first] equality in (v). The second half of the theorem is a consequence of these observations. □

Roughly, Theorem 8.1 resulted from operating on $I + BA$ from the left with an appropriate injective operator. It is reasonable to ask whether there is a counterpart corresponding to operation on the right with a bijective operator that eliminates the appearance of A; in view of Corollary 6.2, one is led to consider a combination of $I + K$ and W_i^2. Thence it appears that there is needed an extension of Ref. 2, Theorem 3.34, the counterpart for the Neumann problem of Ref. 2, Theorem 3.33, pertaining to the Dirichlet problem and arising in, e.g., Ref. 26; the complete references can be found in Ref. 2. Specifically, we have the following result, the proof of which actually employs arguments used in the proof of Ref. 2, Theorem 3.33.

8.2 Lemma

Suppose that $\zeta \in \mathbb{C}$, with $Im\zeta \neq 0$ and $Im\zeta \cdot Re\kappa \geq 0$. Then the operator $I + K + \zeta W_1^2$ (with domain N_1^2) is injective and maps $\mathcal{R}(A)$ onto H_0.

Proof. Suppose that $f \in N_1^2$ and $(I + K + \zeta W_1^2)f = 0$. Set $u_+ := V_+\{f\} + \zeta W_+\{f\}$; then $u_+ \in W(\Omega_+, \kappa)$, while u_+ satisfies (2.3) in Ω_+ and is L_2-regular at Γ. The condition on f shows that $u_+|_1^2 = 0$, whence Corollary 3.5 implies that $u_+ = 0$, and so also $Sf + \zeta(-I + \bar{K}^*)f = u_+|_1^2 = 0$. Thus, for u_-, we get

$$u_-|_1^2 = Sf + \zeta(I+\bar{K}^*)f = 2\zeta f\ ,$$

$$u_-|_1^2 = (-I + K)f + \zeta W_1^2 f = -2f\ .$$

Upon applying the "−" case of (3.8), taking v there as u_- and u as u_-, we find

$$\lim_{\epsilon \to 0^+} \int_{\Omega_-^\epsilon} \{-\kappa^2 |u_-|^2 + |\mathrm{grad}\, u_-|^2\}\, d\lambda = -4\zeta \cdot \int_1 |f|^2\, d\lambda_1\ ,$$

so

$$Im(-\kappa^2) \lim_{\epsilon \to 0^+} \int_{\Omega_-^\epsilon} |u_-|^2\, d\lambda = 4 \cdot Im\zeta \cdot \int_1 |f|^2\, d\lambda_1\ ;$$

just as in the proof of Corollary 3.5, we deduce that $u_- \in L_2(\Omega_-)$, and the limit appearing is just the integral of $|u_-|^2$ over Ω_-. Consequently,

$$-Im\zeta \cdot Re\kappa \cdot Im\kappa \cdot \int_{\Omega_-} |u_-|^2\, d\lambda = 2 \cdot (Im\zeta)^2 \cdot \int_1 |f|^2\, d\lambda_1\ .$$

But $(Im\zeta)^2 > 0$, $Im\zeta \cdot Re\kappa \geq 0$, and $Im\kappa \geq 0$, so $f = 0$. Thus, we have shown that $I + K + \zeta W_1^2$ is injective. Clearly, then, the same assertion concerning $(I + K + \zeta W_1^2)A$ (defined on H_0) is valid (recall that A is injective and $\mathcal{R}(A) \subset N_1^2$), while, with (6.17), we find

$$(I + K + \zeta W_1^2)A = (I + K)A + \zeta(-I + K) = -\zeta\{I - \frac{1}{\zeta}(K + (I + K)A)\}\ .$$

With the compactness of K and A, the Fredholm Theory says that $(I + K + \zeta W_1^2)A$ maps H_0 onto H_0. The proof is complete. □

We shall make a momentary digression to point out a rather intriguing implication of the facts just proven.

8.3 Corollary

$\mathcal{R}(A) = N_1^2$.

Proof. We already know that $\mathcal{R}(A) \subset N_1^2$. Let $f \in N_1^2$; choosing any $\zeta \in \mathbb{C}$ as in the statement of §8.2, we know that there exists (a unique) $f_0 \in \mathcal{R}(A)$ such that $(I + K + \zeta W_1^2)f_0 = (I + K + \zeta W_1^2)f$. $I + K + \zeta W_1^2$ is injective, so $f = f_0 \in \mathcal{R}(A)$.

Returning to the reformulation problem, a second scheme can now be substantiated; one should compare with Refs. 2 and 6.

8.4 Theorem

Let $\zeta \in \mathbb{C}$, with $Im\zeta \neq 0$ and $Im\zeta \cdot Re\kappa \geq 0$. Then

$$I+K+B(S-\zeta(I-\bar{K}^*))+\zeta W_i^2 = (I+BA)(I+K+\zeta W_i^2) \quad \text{on } N_i^2, \tag{8.4}$$

so the operator on the left in (8.4) is injective and maps its domain N_i^2 onto H_o. Further, whenever $g \in H_o$ and $\tilde{f}_g$ is the unique element of N_i^2 such that

$$\{I+K+B(S-\zeta(I-\bar{K}^*))+\zeta W_i^2\}\tilde{f}_g = g, \tag{8.5}$$

then

$$(I+BA)^{-1}g = (I+K+\zeta W_i^2)\tilde{f}_g, \tag{8.6}$$

and

$$A(I+BA)^{-1}g = (S-\zeta(I-\bar{K}^*))\tilde{f}_g. \tag{8.7}$$

Proof. By means of (6.6) and (6.7), for $f \in N_i^2$,

$$(I+BA)(I+K+\zeta W_i^2)f = (I+K)f+BSf+\zeta W_i^2 f+\zeta B(-I+\bar{K}^*)f,$$

verifying (8.4). Now the first statement of the theorem is obtained by recalling Corollary 6.13 and Lemma 8.2. Let $g \in H_o$, and suppose that $\tilde{f}_g$ is the unique element of N_i^2 such that (8.5) holds: then (8.4) implies that (8.6) must hold, and (8.7) is derived, in turn, from (8.6) with the help of (6.6) and (6.7). □

Orthonormalization

Here, we shall derive a representation for the operator $A(I+BA)^{-1}$ that reduces to (6.8) when $B = 0$. For any family $\{v_n\}_{n=1}^{\infty} \subset W(\Omega_+;\kappa)$ as in §5, we have seen in Corollary 6.14 that the corresponding set $\{w_n := v_{n,i}^2+Bv_n|_\Gamma^2 = (I+BA)v_{n,i}^2\}_{n=1}^{\infty}$ is complete in H_o. The latter family is also linearly independent, for, if we should have $\sum_{n=1}^{N} c_n w_n = 0$, then the element $f := \sum_{n=1}^{N} c_n v_{n,i}^2$ of H_o satisfies $(I+BA)f = 0$, and so $f = 0$. With the linear independence of $\{v_{n,i}^2\}_{n=1}^{\infty}$, we must have $c_1 = \cdots = c_N = 0$. Therefore, the Gram-Schmidt procedure can be applied directly to $\{w_n\}_{n=1}^{\infty}$, *i.e.*, we can find $\{b_{nj} \mid j = 1,\ldots,n, n = 1,2,\ldots\}$ such that $\{\hat{w}_n\}_{n=1}^{\infty}$ is an orthonormal basis for H_o, wherein

$$\hat{w}_n := \sum_{j=1}^{n} b_{nj} w_j, \quad n = 1,2,\ldots. \tag{8.8}$$

Defining $\{\tilde{v}_n\}_{n=1}^{\infty} \subset W(\Omega_+;\kappa)$ by

$$\tilde{v}_n := \sum_{j=1}^{n} b_{nj} v_j, \quad n = 1,2,\ldots, \tag{8.9}$$

of course we have

$$\hat{w}_n = (I+BA)\tilde{v}_{n,\nu}^2, \quad n = 1,2,\ldots. \tag{8.10}$$

If $B = 0$, this reduces to the orthonormalization effected in §5, since then $w_n = v_{n,i}^2$ for each n, so we may suppose that $\tilde{v}_n = \hat{v}_n$ for each n in that case. Now, for any $g \in H_o$, we find

$$(I+BA)^{-1}g = (I+BA)^{-1}\sum_{n=1}^{\infty} < g,\hat{w}_n > \hat{w}_n = \sum_{n=1}^{\infty} < g,\hat{w}_n > \tilde{v}_{n,\nu}^2,$$

and so

$$A(I+BA)^{-1}g = \sum_{n=1}^{\infty} < g,\hat{w}_n > \tilde{v}_n|_\Gamma^2.$$

To summarize the development to this point, we have proven:

8.5 Proposition

Let $\{v_n\}_{n=1}^{\infty}$ be any family as in §5.1, and $\{\hat{w}_n\}_{n=1}^{\infty}$ an orthonormal basis for H_o generated from the complete and linearly independent family $\{w_n := v_{n,\nu}^{-} + Bv_n|_\Gamma^{-}\}_{n=1}^{\infty}$ *via* the Gram-Schmidt procedure, so that (8.8) holds. Define $\{\tilde{v}_n\}_{n=1}^{\infty} \subset W(\Omega_+;\kappa)$ as in (8.9). Then

$$(I+BA)^{-1} = \sum_{n=1}^{\infty} < \cdot, \hat{w}_n > \tilde{v}_{n,\nu}^{-} \tag{8.11}$$

and

$$A(I+BA)^{-1} = \sum_{n=1}^{\infty} < \cdot, \hat{w}_n > \tilde{v}_n|_\Gamma^{-}, \tag{8.12}$$

with convergence in the sense of the strong operator topology.

Having the latter statement, one can use (8.12) with the last form of the solution for $ER(g|B;\kappa)$ given in (8.1). Alternately, using the first form in (8.1) and both (8.11) and (8.12), we compute, for any $g \in H_o$ and $x \in \Omega_+$, using the strong convergence,

$$\frac{1}{2}\int_\Gamma \{E_x \cdot (I+BA)^{-1} g - E_{x,\nu} \cdot A(I+BA)^{-1} g\} d\lambda_\Gamma$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\int_\Gamma \{E_x \cdot \tilde{v}_{n,\nu}^{-} - E_{x,\nu} \cdot \tilde{v}_n|_\Gamma^{-}\} d\lambda_\Gamma \cdot < g, \hat{w}_n >$$

$$= \sum_{n=1}^{\infty} < g, \hat{w}_n > \tilde{v}_n(x),$$

the latter equality following from Corollary 3.4. One can proceed in this manner to consider partial derivatives, and error estimates can be derived in a straightforward way, producing a direct proof of the theorem that we are about to state. We shall omit most details of the proof, since the major portion is subsumed by the more general construction through approximation of the boundary data, to which we shall presently turn. The series development given here is a generalization of that derived in Ref. 20 for the classical Neumann problem.

8.6 Theorem

Let $\{\hat{w}_n\}_{n=1}^{\infty} \subset H_o$ and $\{\tilde{v}_n\}_{n=1}^{\infty} \subset W(\Omega_+;\kappa)$ be families constructed as described in §8.5. Let $g \in H_o$. Then the solution function of $ER(g|B;\kappa)$ is given in Ω_+ by

$$u_g := \Phi_{(I+BA)^{-1}g} = \sum_{n=1}^{\infty} < g, \hat{w}_n > \tilde{v}_n, \tag{8.13}$$

the series converging in $W(\Omega_+;\kappa)$. Thus, all partial derivatives of the solution function can be computed *via* term-by-term differentiation of the series in (8.13), and each series so derived, as well as that in (8.13), converges uniformly on each subset of Ω_+ closed in $\mathbf{R}^3$; in fact, if F is any such set and α is any 3-index, then, with $\{M_n(\kappa;F)\}_{n=1}^{\infty}$ as in (6.27),

$$\max_{x \in F} | u_{g,\alpha}(x) - \sum_{j=1}^{n} < g, \hat{w}_j > \tilde{v}_{j,\alpha}(x) |$$

$$\leqslant M_{|\alpha|}(\kappa;F) \cdot \{1+||A||\} \cdot ||(I+BA)^{-1}|| \cdot \{||g||^2 - \sum_{j=1}^{n} |< g, \hat{w}_j >|^2\}^{1/2}, \quad n = 1,2,\ldots . \tag{8.14}$$

Finally, the pointwise convergence in (8.13) and in each differentiated series is absolute.

Proof. The only part of the assertion that will not follow from Theorem 8.8, *infra*, is that concerning the absolute convergence of the series in (8.13) and those obtained from term-by-term differentiation. But, since $(< g, \hat{w}_n >)_{n=1}^{\infty} \in l_2$, this will be a consequence of the succeeding Lemma 8.7.

8.7 Lemma

Let $\{\hat{v}_n\}_{n=1}^{\infty}$ be constructed as in §8.5. Then, whenever α is a 3-index and $x \in \Omega_+$, the sequence $(\hat{v}_{n,\alpha}(x))_{n=1}^{\infty}$ is an element of l_2.

Proof. Choose $x \in \Omega_+$. With (8.10), we get

$$\hat{v}_n(x) = \frac{1}{2}\int_{\Gamma} \{E_x \cdot (I+BA)^{-1}\hat{w}_n - E_{x,\nu} \cdot A(I+BA)^{-1}\hat{w}_n\} d\lambda_1$$

$$= < \hat{w}_n, \frac{1}{2}(I+A^*B^*)^{-1}(\bar{E}_x - A^*\bar{E}_{x,\nu}) >, \quad n = 1,2,\ldots.$$

since $\{\hat{w}_n\}_{n=1}^{\infty}$ is orthonormal in H, the lemma is true for $\alpha = (0,0,0)$. Obviously, one can proceed in a similar manner for the partial derivatives, which can each be computed by differentiating under the integral displayed above.

Boundary Data Approximation

The preceding orthonormalization construction works essentially because we had available a sequence $(u_n)_{n=1}^{\infty}$ in $W(\Omega_+,\kappa)$, *viz.*, the partial sums $(\sum_{i=1}^{n} < g, \hat{w}_i > \hat{v}_i)_{n=1}^{\infty}$, such that $(I+BA)u_n|_{\Gamma} \to g$ in H as $n \to \infty$. Then the sequence $(u_n)_{n=1}^{\infty}$ itself converged in $W(\Omega_+,\kappa)$ to the solution function for $ER(g|B;\kappa)$. That this always happens is a direct consequence of the continuous-dependence estimates in §7.

8.8 Theorem

Let $g \in H$. Let $(u_n)_{n=1}^{\infty}$ be a sequence in $W(\Omega_+,\kappa)$ such that $((I+BA)u_n|_{\Gamma} = u_n|_{\Gamma} + Bu_{n,\nu}|_{\Gamma})_{n=1}^{\infty}$ converges in H, to g. Let u_g denote the solution function for $ER(g|B;\kappa)$. Then $(u_n)_{n=1}^{\infty}$ converges to u_g in $W(\Omega_+,\kappa)$. In fact, if F is a subset of Ω_+ that is closed in $\mathbf{R}^3$, α is a 3-index, and $\{M_n(\kappa,F)\}_{n=1}^{\infty}$ is as in (6.27), then

$$\max_{x \in F} |u_{g,\alpha}(x) - u_{n,\alpha}(x)|$$

$$\leq M_{\alpha}(\kappa,F)\cdot\{1+||A||\}\cdot||(I+BA)^{-1}||\cdot||g-(I+BA)u_n|_{\Gamma}||, \qquad \text{for } n = 1,2,\ldots. \tag{8.15}$$

Proof. Choose a positive integer n. Since u_n is the solution function for $ER((I+BA)u_n|_{\Gamma}|B;\kappa)$, the inequalities in (8.15) result directly from Theorem 7.2. Since $||g-(I+BA)u_n|_{\Gamma}|| \to 0$ as $n \to \infty$, it is clear that the convergence of $(u_n)_{n=1}^{\infty}$ to u_g in $W(\Omega_+,\kappa)$ follows in turn from (8.15). □

Observe that, when $g \in H_0$, if we take $u_n = \sum_{i=1}^{n} < g, \hat{w}_i > \hat{v}_i$, $n = 1,2,\ldots$, then (8.10) shows that

$$||g-(I+BA)u_n|_{\Gamma}|| = ||g - \sum_{i=1}^{n} < g, \hat{w}_i > \hat{w}_i||$$

$$= \{||g||^2 - \sum_{i=1}^{n} |< g, \hat{w}_i >|^2\}^{1/2}, \quad n = 1,2,\ldots.$$

This shows that (8.14) follows from (8.15), essentially completing the proof of Theorem 8.6.

Formulation of a Generalized Moment Problem

The more common terminology for this approach is the "null-field method"; cf., *e.g.*, Refs. 23 and 27, and the originating work of Waterman [28]. The idea underlying this technique consists in the attempt to recapture the unknown field on Γ, in the present case $A(I+BA)^{-1}g$, from the knowledge of its "moments," or inner products, with respect to the elements of a family complete in H_o. Our purpose here is merely to identify the form of such a moment problem appropriate to the generalized exterior Robin problem. Specifically, the desired statement appears as:

8.9 Theorem

As in §5, let $\{v_n\}_{n=1}^{\infty} \subset W(\Omega_e;\kappa)$, with $\{v_{n,\nu}|_\Gamma^2\}_{n=1}^{\infty}$ complete in H_o. Let $g \in H_o$. Then

$$< A(I+BA)^{-1}g,\, v_{n,\nu}|_\Gamma^2 + B^*\bar{v}_n|_\Gamma^2 > \; = \; < g, \bar{v}_n|_\Gamma^2 > , \qquad n = 1, 2, \ldots, \tag{8.16}$$

and, moreover, $A(I+BA)^{-1}g$ is the unique element of H_o possessing the property (8.16).

Proof. By Corollary 6.14, $\{\bar{v}_{n,\nu}|_\Gamma^2 + B^*\bar{v}_n|_\Gamma^2\}_{n=1}^{\infty}$ is complete in H_o, so that there can exist at most one $f \in H_o$ such that $< f, v_{n,\nu}|_\Gamma^2 + B^* v_n|_\Gamma^2 > \; = \; < g, \bar{v}_n|_\Gamma^2 >$ for each n. On the other hand, we compute, recalling Corollary 6.5,

$$
\begin{aligned}
< A(I+BA)^{-1}g,\, & v_{n,\nu}|_\Gamma^2 + B^*\bar{v}_n|_\Gamma^2 > \\
&= \; < A(I+BA)^{-1}g, (I+B^*A^*)\bar{v}_{n,\nu}|_\Gamma^2 > \\
&= \; < g, (I+BA)^{-1*}A^*(I+B^*A^*)\bar{v}_{n,\nu}|_\Gamma^2 > \\
&= \; < g, (I+A^*B^*)^{-1}(I+A^*B^*)A^*\bar{v}_{n,\nu}|_\Gamma^2 > \\
&= \; < g, \bar{v}_n|_\Gamma^2 > , \qquad n = 1, 2, \ldots . \; \Box
\end{aligned}
$$

Thus, the unknown field $A(I+BA)^{-1}g$ is completely characterized by the equalities in (8.16), even if, say, $B = 0$ and κ^2 is a Dirichlet eigenvalue for $-\Delta$ in Ω. Therefore, the reformulation expressed by Theorem 8.9, like the preceding ones, can afford an algorithm that will be valid uniformly in κ, i.e., which is free of the need for *a priori* knowledge of values of κ at which the numerical scheme will fail.

In practice, the conclusion of Theorem 8.9 for the Neumann problem (when $B = 0$) has been formally exploited by selecting some family $\{v_n^*\}_{n=1}^{\infty} \subset H_o$, choosing a positive integer N, assuming an approximation for Ag of the form $\sum_{n=1}^{N} c_n^N v_n^*$, and determining the coefficients $\{c_n^N\}_{n=1}^{N}$ so that the first N equalities in (8.16) hold when Ag is replaced by the sum (assuming that the pertinent N-by-N matrix of inner products is nonsingular). Evidently, there have been given to this date in the literature few satisfactory analyses of the questions of convergence associated with such a scheme, either in the form of general criteria to be fulfilled by the family $\{v_n^*\}_{n=1}^{\infty}$ or for a specific such family. No attempt to elucidate these matters will be made here.

9. IMPLICATIONS FOR THE DIRICHLET PROBLEM

Let us consider what can be said concerning

9.1 The Exterior Dirichlet Problem ED($g;\kappa$):

Let $g \in H_o$; show that there exists precisely one $v_g \in W(\Omega_e;\kappa)$ such that

$$v_g|_\Gamma^2 = g. \tag{9.1}$$

Of course, Corollary 3.5 implies that $ED(0|\kappa)$ has only the trivial solution, but otherwise the state of affairs here is not nearly so nice as for the Robin problem. Evidently, $\mathcal{W}(\Omega_e,\kappa)$ is the "wrong" space in which to pose the Dirichlet problem if one insists on requiring that the boundary condition be fulfilled in the L_2-normal-trace sense. Put another way, it is too much to expect that there exist a solution of $ED(g|\kappa)$ for each $g \in H_0$ that also has a normal derivative on Γ in the L_2-sense. The next statement clarifies these remarks and shows that the difficulty stems from the compactness of A.

9.2 Theorem

$ED(g|\kappa)$ is solvable *iff* $g \in N_e^2 = \mathcal{R}(A)$, in which case it is solved *via* the function $v_g = \Phi_{A^{-1}g}$, *i.e.*, by $\frac{1}{2}\{\mathcal{V}_e\{A^{-1}g\} - \mathcal{W}_e\{g\}\}$.

Proof. Let $g \in H_0$. As remarked, it is clear that $ED(g|\kappa)$ can possess at most one solution function. If $g \in \mathcal{R}(A)$, then Theorem 6.9 shows that $v_g = \Phi_{A^{-1}g}$ is in $\mathcal{W}(\Omega_e,\kappa)$, with $v_g|_\Gamma^2 = AA^{-1}g = g$ (and $v_g|_\Gamma^1 = A^{-1}g$). Thus, $\mathcal{R}(A) \subset \{v|_\Gamma^2 \mid v \in \mathcal{W}(\Omega_e,\kappa)\}$, while the opposite inclusion between the latter sets is obvious from Lemma 6.1, whence $\mathcal{R}(A) = \{v|_\Gamma^2 \mid v \in \mathcal{W}(\Omega_e,\kappa)\}$. With these observations, the proof is complete. □

Since A is compact and injective, we know that $\mathcal{R}(A)$ is properly contained in H_0 (although it is dense in H_0, since A^* is injective), while A^{-1} is unbounded on $\mathcal{R}(A)$. Thus, in view of §9.2, we are sure that $ED(g|\kappa)$ cannot be solved for every $g \in H_0$. Moreover, to achieve continuity of the solution map $g \to v_g$ on $\mathcal{R}(A)$ into $\mathcal{W}(\Omega_e,\kappa)$ (the latter having the locally convex linear topological space structure described in §6), evidently we must equip $\mathcal{R}(A)$ with the graph norm induced by A^{-1}, $\|g\|_{A^{-1}} = \{\|g\|^2 + \|A^{-1}g\|^2\}^{1/2}$, $g \in \mathcal{R}(A)$, under which $\mathcal{R}(A)$ is a Hilbert space. Then, by replacing H_0 in §9.1 by $\mathcal{R}(A)$, we should arrive at a well-posed problem. Nevertheless, this merely skirts the basic question concerning the existence of a solution of the Dirichlet problem for any boundary data chosen from H_0. For this, we must enlarge the set in which solutions are sought, posing the problem as:

> Let $g \in H_0$; show that there exists precisely one $v_g \in C^2(\Omega_e)$ that satisfies (2.3) in Ω_e, fulfills condition (2.4), and possesses a normal trace on Γ in the L_2-sense, with $v_g|_\Gamma^2 = g$.

We shall leave open the resolution of this problem.

10. EXTENSIONS

The essential properties of the operator A, in particular Theorem 6.9, provide a means for attacking the study of exterior boundary-value problems for the Helmholtz equation under boundary conditions much more general than that expressed by (2.6). Indeed, suppose that there is given a mapping $\Psi: H_0 \times H_0 \to H_0$ (which need not be linear in either of its arguments), and consider the problem of showing that there exists $w \in \mathcal{W}(\Omega_e,\kappa)$ such that

$$\Psi(w|_\Gamma^2,\ w|_\Gamma^1) = g\,, \tag{10.1}$$

or

$$\Psi(w|_\Gamma^2,\ Aw|_\Gamma^2) = g\,, \tag{10.2}$$

in which $g \in H_0$. If w is a solution, then $w = \Phi_{w|_\Gamma^2}$ (cf. 6.9.ii) and $w|_\Gamma^2$ satisfies (10.2). On the other hand, if $h \in H_0$ satisfies

$$\Psi(h, Ah) = g\,, \tag{10.3}$$

then Φ_h, as in (6.19) is clearly a solution of the problem posed. Consequently, there exist precisely as many solutions of the boundary-value problem corresponding to $g \in H_0$ as there exist solutions of the operator problem in (10.3). In this way, one is led to examine the map $h \to \Psi(h, Ah)$ on H_0 into H_0.

in particular, aiming at identifying its range and the number of solutions of (10.3) for each g in that range. Of course, the Robin problem §2.4 corresponds to the choice $(f_1, f_2) \to f_1 + Bf_2$ for Ψ, and (10.3) takes the form $(I+BA)h = g$; the interplay between the dissipative property of $i\zeta_\kappa^H B$ and the strictly dissipative property of $-i\zeta_\kappa^H A$ yielded, in Corollary 6.13, the bijective property of $I+BA$, whence Theorem 7.1 followed. Incidentally, these remarks provide another method for proving the uniqueness assertion in Theorem 7.1, *i.e.*, we need not have stated Corollary 3.6. The crucial fact is contained in Corollary 3.5, leading to §6.13 through §6.10.

As a simple example immediately suggesting itself, let $\Psi_o : H_o \to H_o$ be Lipschitz continuous, so that, for some $l_o > 0$,

$$\| \Psi_o(f_2) - \Psi_o(f_1) \| \leqslant l_o \cdot \| f_2 - f_1 \|, \qquad f_1, f_2 \in H_o .$$

Let us suppose further that $l_o \cdot \| A \| < 1$. Then clearly $\Psi_o \circ A$ is a contraction on H_o, so there exists, for each $g \in H_o$ a unique $h_g \in H_o$ with $h_g + \Psi_o(Ah_g) = g$. Thus, there is in this case a unique w_g in $\mathcal{H}(\Omega_-;\kappa)$, *viz.*, Φ_{h_g}, satisfying the boundary condition

$$w_{g,\nu}|_\Gamma^2 + \Psi_o(w_g|_\Gamma^2) = g,$$

for each $g \in H_o$.

Aside from these indications for the treatment of nonlinear boundary conditions, the observations of this section have further significance for the problem posed in §2.4. In fact, the proof of the following result is nearly immediate.

10.1 Theorem

Let $B_\pi : H_o \to H_o$ be any bounded linear ("perturbation") operator with $\| B_\pi \| < \| A \|^{-1} \cdot \| (I+BA)^{-1} \|^{-1}$; B satisfies (2.2), as always, but B_π need not. Then $I+(B+B_\pi)A$ possesses a bounded inverse on H_o, so, for each $h \in H_o$, the function $u_h^\pi := \Phi_{(I+(B+B_\pi)A)^{-1}h}$ is the unique element of $\mathcal{H}(\Omega_-;\kappa)$ satisfying the boundary condition

$$u_{h,\nu}^{\pi}|_\Gamma^2 + (B+B_\pi)\, u_h^\pi|_\Gamma^2 = h.$$

Moreover, for each $F \subset \Omega_-$ closed in $\mathbb{R}^3$, with u_g denoting the solution of $ER(g|B;\kappa)$,

$$\max_{x \in F} | u_{g,\alpha}(x) - u_{h,\alpha}^\pi(x) |$$

$$\leqslant M_{|\alpha|}(\kappa;F) \cdot \{1+\| A \|\} \cdot \{ \|(I+BA)^{-1}\| \cdot \|g-h\|$$

$$+ \frac{\| A \| \cdot \| (I+BA)^{-1} \|^2 \cdot \| B_\pi \|}{1 - \| A \| \cdot \| (I+BA)^{-1} \| \cdot \| B_\pi \|} \cdot \| h \| \}, \tag{10.4}$$

holding for $g, h \in H_o$ and any 3-index α, with $\{M_n(\kappa;F)\}_{n=1}^\infty$ as in (6.27).

Proof. Since $I+(B+B_\pi)A = (I+BA)\{I+(I+BA)^{-1}B_\pi A\}$, it is clear from the condition placed upon $\| B_\pi \|$ that $I+(B+B_\pi)A$ has a bounded inverse defined on H_o. The first part of the theorem now holds in virtue of the remarks made in the first paragraph of this section. The inequalities in (10.4) are proven by following the steps that yielded those in (7.1). □

Theorem 10.1 shows that the errors in predictions made on the basis of approximations to B and g will be within prescribed tolerances if the approximations are sufficiently accurate.

One can now readily verify that the completeness results and construction techniques given earlier for B can be extended to statements concerning $B+B_\pi$, with B_π as in Theorem 10.1; for example, $\{|v_{n,s}|^2 + (B+B_\pi)\, v_n|_\Gamma^2\}_{n=1}^{\infty}$ is complete in H_α.

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